INTEGRAL EQUATIONS IN ELASTICITY

V. Z. Parton and P. I. Perlin

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ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ ТЕОРИИ УПРУГОСТИ

ИЗДАТЕЛЬСТВО «НАУКА» • МОСКВА

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MIR PUBLISHERS
MOSCOW

Translated from the Russian

First published 1982
Revised from the 1977 Russian edition

На английском языке

Contents

Preface to the English Edition

The classical investigations of Fredholm, Hilbert, Poincaré, Muskhelishvili, Tricomi, Giraud, Mikhlin on the theory of integral equations (one-dimensional and multidimensional, regular and singular) determined the success of their application to the solution of boundary value problems in mathematical physics, and in particular in elasticity.

As is known, only in exceptional cases are the integral equations amenable to analytic solution, and their use therefore became possible only with the advent of high-speed computer facilities.

Taking into account the increasing interest of specialists, engaged in the elasticity theory and its various applications, in the fundamentals of the theory of integral equations and in the substantiation and implementation of particular computational algorithms, the authors have made an attempt to present the most important achievements in a form intelligible to scientists, engineers, postgraduates and undergraduates.

Compared to the Russian edition (1977), this edition has been revised mainly to include new results. Sections 17, 25, 33, 37 have been completely revised, and some alterations have been introduced into others. Much valuable advice has been offered recently by Prof. V. G. Maz'ya.

The authors consider it a great honour to have the book published in English and express their sincere graditude to Acad. O. M. Belotserkovskii, Acad. I. F. Obraztsov and Acad. L. I. Sedov for their help in this.

V. Z. Parton
P. I. Perlin

Moscow
July, 1980

Preface to the Russian Edition

The rapid development of computer engineering has aroused the considerable interest of researchers for the development of universal numerical methods for the solution of elasticity problems. Together with the generally accepted methods in continuum mechanics and engineering calculations such as the finite element method and the finite difference method, one of the most promising and efficient methods is the potential method, which reduces boundary value problems to corresponding integral equations. The chief advantage of this method is that it reduces the dimensionality of problems considered.

This book presents the fundamentals of the theory of regular and singular integral equations in the case of one and two variables. The general principles of the theory of approximate methods are considered as well as their application for the efficient solution of both regular and singular integral equations. The necessary information is given on the three-dimensional and two-dimensional equations of the theory of elasticity including the formulation of boundary value problems. The book contains the derivation and analysis of various integral equations of the plane problem for both fundamental boundary value problems and mixed problems, and also for bodies with cuts. In the three-dimensional case the construction and analysis of integral equations are carried out for the first and second fundamental problems.

Emphasis is placed on efficient methods for solving integral equations for the plane and three-dimensional problems of elasticity. Examples are given illustrating the advantages of a particular approach. The book is appended with an extensive list of references giving comprehensive information of the subject of investigation.

The emphasis on numerical methods for the solution of integral equations for elastostatic problems corresponds to the author's conviction that this approach has considerable promise, particularly with the advent of the nearest-generation computers.

The scope of the book is limited to elastostatic problems though the extension of the methods described to dynamic problems apparently involves no fundamental difficulties.

The interested reader can find a more detailed mathematical presentation of the theory of integral equations in the well-known books by F. D. Gakhov, V. D. Kupradze, S. G. Mikhlin, and N. I. Muskhelishvili.

S. G. Mikhlin and D. I. Sherman have rendered the authors great assistance in preparing the book. I. I. Vorovich and V. V. Panasyuk have given much useful advice.

The book owes its appearance to a large extent to the care and help in preparing the manuscript by T. A. Alikhanova.

The authors take the opportunity of expressing to all these colleagues their sincere gratitude.

V. Z. Parton
P. I. Perlin

On the Formation of Integral Equation Methods in the Theory of Elasticity

The intensive development of computer engineering proceeds in tandom with ever increasing interaction and stimulation, with the development of universal methods of solving problems in mathematical physics; among these we should first of all note the finite element method and the finite difference method, which have found wide application in engineering calculations. However, problems of increased difficulty, among which are naturally the fundamental three-dimensional problems of elasticity and also two-dimensional problems for regions of rather complex configuration, are often almost impossible to solve at present by the above methods with the required degree of accuracy. At the same time these methods must be given full credit; because of their simplicity, they are easy to understand and intelligible to engineers, and in future the development of these methods will invariably, and to a progressively larger extent, accompany the steady improvement of computers.

At present, in difficult cases preference is given to other methods, and among these is the most generally used integral equation method (both regular and singular equations).

We remember very well the time when almost every author dealing with a non-trivial elasticity problem considered it very nearly a matter of his honour to reduce it by all means to a Fredholm equation of the second kind. After this, he was prone at least to think that his investigation was completed theoretically without concerning himself with the implementation of the solution. (People wonder at it now.)

In the complex, and in general multistage, procedure of reducing various problems of mathematical physics to solvable Fredholm integral equations of the second kind, the role of one of the main links should be attributed to the construction of the so-called fundamental (auxiliary) solution, namely a function (or matrix) depending on two points and satisfying the original differential equation (or system of equations) for the co-ordinates of at least one of these points.

The technique of constructing fundamental solutions is no secret and to a certain extent is advanced. Meanwhile, fundamental solutions are sometimes (mainly in three-dimensional problems) so lengthy and unwieldy that there can be no question of using them directly for their proper purpose. There is an imperative need to simplify the structure of the fundamental solution so as to render it more or less manageable. In so doing, it is necessary to keep carefully its principal part unaltered, which, in fact, determines the reduction of the problem in hand to a Fredholm equation of the second kind. Naturally, the resulting final form (very curtailed) of the original fundamental solution is no longer such in the accepted sense of the term, nevertheless we will retain this name for it.

One next proceeds to the concluding phase, i.e., to the construction of a proper representation for the unknown function (vector function); it is chosen in the form of an integral taken over the boundary of a body and containing under its sign the product of the fundamental solution and a certain function (not yet known) termed the density function. The representation thus constructed is commonly termed the potential.

As is known, the simplest potentials, so-called simple-layer and double-layer potentials, are quite often used in many problems, particularly in classical harmonic problems. These potentials have been thoroughly and scrupulously studied in the fundamental works of A. M. Lyapunov for a very wide class of surfaces, known by his name.

Having at our disposal the expression for the elastic potential thus developed, and proceeding to the boundary conditions of the problem, we obtain a Fredholm integral equation of the second kind for determining the density. The next, and no less serious, concern to come to the forefront is the vital question of the solvability of the equation obtained. It is not always possible to give a definite answer at once; this as a rule involves specific, and sometimes considerable, difficulties. If contrary to expectations based on some, not very solid, considerations it is found that the integral equation under study is unsolvable, then new complications arise; they can be eliminated in different ways, or, more precisely, in two ways, if at all. On the one hand, attempts may be made to modify the representation in such a way to obtain a true representability of the desired solution by means of the final potential. If this route is found to be excessively difficult and ineffectual, it might be worthwhile trying to modify the equation itself by introducing certain, possibly elementary, operators depending on the unknown density; the operators must be deliberately chosen so that, without changing the main point (i.e., without deviating from the stated problem), the eigenfunctions inherent in the integral equation are eliminated. Sometimes the investigator is compelled to resort to both procedures.

It is precisely this approach that has led to a positive result in discussing the first fundamental three-dimensional problem of the theory of elasticity.

Of course, this kind of modernization of the integral equation is not at all simple and often requires prolonged thought and repeated trials; this is almost as difficult as, and sometimes even more difficult than, the fulfilment of the reconstruction of the fundamental solution; in general, neither, even with wide experience in investigation of this kind, can be realized without much effort; unfortunately, the efforts are not always successful.

Let us briefly consider the current status of the theory of integral equations in reference to the fundamental three-dimensional elastostatic problems, a subject of vital importance to application and covered in considerable detail in the book.

Soon after E. I. Fredholm developed the theory of integral equations of the second kind, called by his name, they were applied successfully in the investigation of the Dirichlet and Neumann boundary value problems,* many attempts have been made to carry out a similar analysis in the case of fundamental problems of the theory of elasticity. For the first fundamental three-dimensional problem (with given displacements), this was done by G. Lauricella in 1907 by choosing a suitable fundamental solution. On the face of it, this choice seems to be rather artificial; actually, it is made in a more or less natural way and on the basis of clear considerations. As shown by Lauricella himself, his equations are always solvable for any finite body bounded by a single surface. With a slight modification of these equations, noted above, they become solvable both for a finite and an infinite body and in the case of several bounding surfaces.

Unfortunately, the construction of regular integral equations for the three-dimensional problem with given external stresses is much more difficult. In 1915 H. Weyl constructed relatively simple Fredholm equations for this problem in the case of a finite region bounded by one closed convex surface. Up to now, however, they are still not well enough ordered to permit the unreserved application of the available computational algorithms to them. Further the same author showed how the resulting equations could be modified to make them suitable, in principle, for an arbitrary three-dimensional region. Unfortunately, this method is not simple, involving recourse to Green's tensor and leading to unwieldy equations, scarcely amenable either to direct examination or to use for their proper purpose.

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* Note that for the most general case when bodies are bounded by several Lyapunov surfaces, a profound and comprehensive study of such equations was made in the works of N. M. Günter.

At one time it seemed probable to many, at least at first sight, that the reduction of boundary value problems of elasticity to regular integral equations could be achieved in a simpler and more vivid way by taking for fundamental solutions very elementary formations. These were most often taken as the solutions of a three-dimensional problem involving a concentrated force and a concentrated moment. However, starting from these auxiliary solutions, many were soon disappointed since the equations thus obtained were irregular and fell out of the Fredholm equation class. This circumstance had a frightening effect on the great majority of initiators, and they soon retreated. With a considerable delay it must now be admitted that, to general satisfaction, there were more shrewd investigators who were not daunted by unknown difficulties and a vague perspective and devoted much time to a thorough study of the equations in question. As a result, a rigorous theory of so-called singular integral equations of one or more dimensions has been developed, which has been widely accepted and has crystallized into an independent section of the general theory of integral equations. Equations of this kind have been constructed and investigated scrupulously in reference to fundamental problems of elasticity. The strictly established fact of the applicability of the Fredholm alternatives to these equations has aided appreciably in gaining a deep insight into their structure and, moreover, in revealing a number of their important spectral properties. After this, the possibility of using these equations in practice became quite obvious.

It should be noted that the singular equations as such have long attracted attention; these equations were treated by such brilliant mathematicians as Poincaré and Hilbert; they were primarily concerned with the case when the kernel is of the form of the cotangent function. Later the general theory of such equations was constructed by F. Noether.

It might be as well to point out the exceptional role of the excellent work of T. Carleman (1922) in the development of the theory of singular equations with a Cauchy kernel. It has won great popularity among theorists and has also found favour among a wide circle of engineers. Carleman was the first to give a solution to an important type of singular integral equation in closed form with wonderful elegance and simplicity, and this opened the way for the efficient treatment of a wide class of mixed problems of mathematical physics.

N. I. Muskhelishvili, as well as other prominent investigators, has made a significant contribution to the development of the theory of one-dimensional singular equations and its applications in various problems of mathematical physics.

Nevertheless, in the beginning the singular equations could frighten many, and particularly, of course, many of the engineers who first encountered them. However, by gradually appreciating and

becoming familiar with the nature of their properties, they began not only to get used to these equations, but also to realize how wide a range of questions of mathematical physics they covered. Naturally, all this was acquired and mastered not immediately, but only with time, after prolonged and strenuous reflections. It should be emphasized that an analysis of some selected elasticity problems by using Fredholm equations of the second kind, even though specially adapted and skilfully constructed, may sometimes be technically a somewhat more difficult task than would be the case if singular equations that suggest themselves for the same problems were used. (To avoid misunderstanding, it might be well to point out that examples can easily be given illustrating the reverse.)

We shall mention one more circumstance adding to the incredulity (and not only of engineers) with regard to singular integral equations. The integral equation technique by its nature provides, as a rule, an approximate numerical implementation. However, the question of evaluating singular integrals, especially two-dimensional, which cannot but be decisive when applying certain algorithms, was left open until recently. Moreover, very serious, fundamental corrections had to be introduced into the well-known proofs of convergence for particular methods of approximate solution previously applied to regular equations.

It might be as well to recall that the incipient change in the assessment of the outlook for singular integral equations was properly reflected in our review paper [see *Trudy Vsesoyuznogo S'ezda po Teoreticheskoi i Prikladnoi Mekhanike*, Izd. Akad. Nauk SSSR, 1962 (*Proceedings of the All-Union Congress on Theoretical and Applied Mechanics*, USSR Acad. Sc.)].

As regards the fundamental two-dimensional (plane) problems of the theory of elasticity, the situation was naturally much more favourable here. In the case of basic problems various authors obtained regular integral equations. In due course they were thoroughly studied and still find fairly wide applications.

Taking the opportunity, we emphasize the universally known services of G.V. Kolosov to the development of the theory of elasticity; his works, as well as those of N. I. Muskhelishvili, underlie the specialized and ramified techniques created by many scientists to treat various complex problems of two-dimensional elasticity.

Among the most ponderable achievements is the work of V. A. Fock (1926) who first applied conformal mapping to reduce the plane problem for a simply connected finite region to a Fredholm equation of the second kind (earlier Lauricella had directly reduced the same problem to a Fredholm equation of a different structure without recourse to conformal mapping).

Naturally, the approaches developed for constructing singular integral equations for three-dimensional problems could be adopted

for plane problems, which led to the corresponding one-dimensional singular equations. Note, also, that singular integral equations are constructed without much difficulty in the case of problems with mixed boundary conditions and for bodies with curvilinear cuts of arbitrary configuration.

Concurrent with the investigations mentioned above and up to the present time studies have been carried out intensively on special classes of problems of three-dimensional elasticity; but they are also based on the application of integral equations in one form or another. By way of example, one cannot but mention a great many works dealing with the reduction of axially symmetric problems of elasticity to regular or singular integral equations by superimposing plane states; along with the well-known propositions of the theory of analytic functions use is made of specialized ones. Later this method made it possible to proceed to the consideration of problems for bodies of revolution under non-axisymmetric loading as well. In a number of problems for some bodies of revolution subjected to loads of a special type, it was possible to obtain solutions in closed form (in quadratures). Wider classes of bodies, namely those generated by revolving a closed contour, were considered recently. Finally, it should be noted that the approach under discussion, as one constructed on the basis of an intelligible and legitimate physical concept, by its very nature must be fairly promising.

Besides the foregoing method, there exist other, no less interesting, variants, developed by G. N. Polozhii, of using functions of a complex variable in the same axially symmetric problems of elasticity. The procedures of reducing them to Fredholm equations (in no way based on auxiliary considerations of a mechanical engineering kind) are of a purely mathematical nature and use some classes of generalized analytic functions as the base; their properties have been thoroughly studied by the author himself.

We shall also mention works in which the integral equations for the axially symmetric problem are constructed by suitably adapting the integral equations for the general (three-dimensional) problem and also by using specially introduced axially symmetric potentials.

The facts and findings accumulated throughout the strenuous work outlined above, successively analysed and then summed up and synthesized are, taken together, a factor of increasing significance; on the one hand it will promote an invariable improvement and renewal of numerical algorithms, and on the other hand it will contribute appreciably to the evolution of the potentialities of computers.

In the proposed book the authors set forth the fundamentals both of the general theory of integral equations, regular and singular, and of the theory of approximate methods for their solution, then proceed to the consideration of the integral equations for the plane

and three-dimensional problems of the theory of elasticity, and complete the investigation by an analysis of various efficient methods of solution.

It was very sensible and far-sighted of the authors to propose a considerably simplified presentation of the material (primarily concerning the theory of two-dimensional singular integral equations and its applications); at the same time the authors try to convey information with precision and clarity and are apparently very careful not to permit any elements of vulgarization. Such presentation is adopted in an understandable effort to make the book intelligible to a possibly wider circle of scientists and engineers, and thereby to promote more extensive use of the potential method. This, in turn, will probably have a beneficial effect on the range of applicability of the classical theory of elasticity in applied branches. Those readers who upon acquaintance with the present book feel an increased interest in the subject under investigation and wish to extend and deepen their knowledge on it may refer to the available monographs by F. D. Gakhov, V. D. Kupradze, S. G. Mikhlin, N. I. Muskhelishvili, A. N. Tikhonov, and V. Ya. Arsenin; each of the books is remarkable in its own way.

Following the adopted dominating idea permeating through the entire book, the authors, in full accord with it, strictly and persistently orientate the reader towards advantages derived from a rational use of numerical methods in solving integral equations of elasticity for a sufficiently wide class of problems. This is demonstrated unambiguously by a thorough and purposeful selection of the material included in the book entirely relating to commentary on a series of classical problems of the theory of elasticity; the latter appear as a fully suitable and favourable object for demonstrating the efficiency of numerical analysis (of course, using computers). Incidentally, the selection of material has been made from a great variety of problems and associated methods, in short, from all that is at the disposal of the modern mathematical theory of elasticity.

It might be well to point out the significance of the original works of the authors themselves represented in the book. Of great importance is direct recourse to regular representations of singular integrals, which now makes it possible to use without difficulty the well-known cubature formulas established for improper integrals; it is worth mentioning the advanced methods for the efficient solution of a special class of integral equations by properly applied approximate factorization. The book strongly advertises the procedure of solving integral equations for three-dimensional problems by successive approximations (in view of obvious advantages in the case of implementation on computers). It is established that (in principle) this process is always convergent. However, in solving the second interior problem it is shown that the error in the computational

scheme (because of the presence of an eigenvalue) may, in general, lead to the divergence of the algorithm. To eliminate this discrepancy, it is suggested (as far as we are aware, for the first time) that a specially chosen small additional quantity expressed in a certain manner in terms of the known fundamental functions of the companion equation should be introduced into the right-hand side at each subsequent iteration. This operation outwardly appears to be quite ordinary; however, to hit upon it, as is usually the case under such circumstances, has apparently been possible only after many attempts and prolonged reflections.

In our opinion, there is every reason to believe that the appearance of this book will be received with interest and will evoke a warm response. Because of the simplicity of presentation, the reader can, if he wishes, gain some insight into the problems covered without excessive loss of time. Moreover, and this is the chief thing, the book is a valuable aid for young scientific workers (including those actively engaged in applications) on a difficult road of gaining a fundamental understanding and a thorough mastering of various computational algorithms. This matter brooks no delay; otherwise these algorithms cannot be used to best advantage and their further development is impossible. All that we have noted with regard to the distinguishing features of the book justifies, we believe, its publication.

The foregoing will be concluded by the following remark, in our opinion very timely and necessary. The scientific and technical progress whose witnesses we have the good luck to be involves nearly all fields of knowledge and application. The necessity to keep abreast with the times in this situation considerably increases the sense of responsibility of each scientific worker. Every investigator in the field of the mathematical theory of elasticity and, moreover, in each section of mathematical physics must, in these conditions, critically reconsider his results, emphasize the aspects of his investigations that, as before, need be promoted intensively and reject boldly, without hesitation the part that has lost its effectiveness and cannot be compared with newly developed works. Few people can carry out such an act painlessly; it is usually preceded by agonizing doubts and prolonged thought. Of course, here it is a matter not of the sense of false pride that is hard to overcome, but of other, much more important and quite understandable reasons. After all, the question we have touched upon, painful as it is, is sometimes virtually inseparable from the appraisal of the total work of a scientist. In this case it at times acquires a special acuteness and an undisguised dramatic nature. The same question does not become any simpler in a less radical formulation. Psychologically it is incredibly difficult to renounce even partial results that have taken much intellectual effort during a more or less long period of time. However, obeying

the call of time, an investigator must be able to muster sufficient strength to fulfil the demands of the times by overcoming the habitual predilections and inclinations, which, it would seem, cannot be given up. The dialectics of life is inexorable.

Incidentally, one must not linger and see when somebody else, on his own initiative, undertakes the realization of the urgent task (in respect to one of us) and, in all probability, does it without the proper discretion and thoughtfulness necessary in such delicate circumstances; it is quite possible that the excessive zeal in fulfilling the mission assigned to himself of his own free will may lead him inadvertently to the belittling of even the positive part of the work being revised.

In this connection a question naturally arises as to whether there is sufficient reason for sentiments of gloom and disappointment ostensibly generated by the present situation. In our opinion, there is as a rule no valid reason for this. Of course, few people can boast of the abundance of brilliant ideas visiting them. This is the lot of the minions of fortune generously endowed by nature. For the rest useful notions occur to zealous workers constantly engrossed in thought and selflessly devoted to work. Such people constitute the overwhelming majority. A guess once made, subsequently developed and clearly formulated is hardly ever completely transient; along with other similar guesses it makes a feasible contribution to the fund of gradually accumulated partial results and information paving the way for a long-expected decisive qualitative leap.

All that was done in the past and the present will continue to live in one way or another (and possibly in a modified form). The threads of the past rarely break, they almost invariably extend into the future. The immutability of this self-apparent statement is usually confirmed by the course of the development of scientific thought. It is in this that each of us must and has the right to derive satisfaction and a moral stimulus for his own work.

D. I. Sherman

Notation

Matrices, vectors, and operators are given in bold-faced type.

λ and μ	Lamé's coefficients
$E = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$	elastic modulus
$\nu = \frac{\lambda}{2(\lambda+\mu)}$	Poisson's ratio
D	region (three-dimensional or plane)
S	surface
L	contour
p	point of a (three-dimensional or plane) region
$z = x + iy$	point of a region (plane)
p, q, q'	points of a surface
q, t, τ	points of a contour
$\mathbf{n}\ (n_x, n_y, n_z)$	unit normal to a surface, usually outward (if the surface is closed)
$\mathbf{T}_{n(p)}$	stress operator
$\mathbf{N}_{n(p)}$	N-operator
$\mathbf{\Gamma}(p, q)$	Kelvin-Somigliana matrix (kernel of a simple-layer potential)
$\mathbf{\Gamma}_2^{\mathrm{I}}(p, q)$	matrix (kernel of a double-layer potential of the first kind)
$\mathbf{\Gamma}_2^{\mathrm{II}}(p, q)$	matrix (kernel of a double-layer potential of the second kind)
$\mathbf{\Gamma}_1(p, q)$	matrix (the result of the action of the stress operator on the matrix $\mathbf{\Gamma}(p, q)$, i.e., $\mathbf{\Gamma}_1(p, q) = \mathbf{T}_{n(p)}\mathbf{\Gamma}(p, q)$
$\boldsymbol{\varphi}(q)\ (\varphi_1(q), \varphi_2(q), \varphi_3(q))$	density of potentials
$\mathbf{V}(p) = \mathbf{V}(p, \boldsymbol{\varphi})$	symbolic form of writing a simple-layer potential with density $\boldsymbol{\varphi}$

$W(p) = W(p, \varphi)$	symbolic form of writing a double-layer potential with density φ
$f(q, \psi)$	characteristic of a singular integral
$\Phi(q, \lambda)$	symbol of a singular operator
K, $\tilde{K}$, A	general form of writing operators
$z = \omega(\zeta)$	function effecting conformal mapping
$P_n(x)$	Legendre polynomial of order n
$P_n^{(\alpha, \beta)}(x)$	Jacobi polynomial of order n
$K(x, y)$, $\Gamma_1(q, q')$, $\Gamma_2(q, q')$	kernels of integral equations
$\Gamma(x, y, \lambda)$	resolvent of an integral equation

Three-dimensional problem

x_1, x_2, x_3 (y_1, y_2, y_3)	co-ordinates in a Cartesian system
σ_{ij}	components of the stress tensor
$u(u_1, u_2, u_3)$	displacement vector
ε_{ij}	components of the strain tensor

Two-dimensional problem

x, y, z	co-ordinates in a Cartesian system
σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}	components of the stress tensor
u, v, w	components of the displacement vector

Chapter I

ELEMENTS OF THE THEORY OF ONE-DIMENSIONAL AND MULTIDIMENSIONAL INTEGRAL EQUATIONS

1. Analytic Theory of a Resolvent

Consider a Fredholm integral equation of the second kind. For simplicity of presentation, we restrict our study to the case of one dimension:

$$\varphi(x) = \lambda \int_a^b K(x, y)\, \varphi(y)\, dy + F(x). \tag{1.1}$$

The kernel $K(x, y)$ is assumed to be an integrable function of x and y $(a \leqslant x \leqslant b,\ a \leqslant y \leqslant b)$ and the solution of Eq. (1.1) is sought in the form of a series:

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \ldots \tag{1.2}$$

If the series converges uniformly for some values of the parameter λ, it may be substituted in Eq. (1.1); by equating the coefficients of like powers of λ, we obtain the recurrence relations

$$\varphi_i(x) = \int_a^b K(x, y)\, \varphi_{i-1}(y)\, dy \quad (i = 1, 2, \ldots), \tag{1.3}$$

$$\varphi_0(x) = F(x).$$

Suppose that the kernel $K(x, y)$ and the function $F(x)$ are bounded $(|K(x, y)| < A,\ |F(x)| < M)$. It follows from relations (1.3) that the unknown solution (1.2) is majorized by the series

$$M \sum_{n=0}^{\infty} |\lambda|^n (b-a)^n A^n,$$

Consequently, series (1.2) converges if

$$|\lambda| < \frac{1}{A(b-a)}. \tag{1.4}$$

By using relations (1.3), we obtain a different representation of series (1.2):

$$\varphi(x) = F(x) + \lambda \left\{ \int_a^b K_1(x, y) F(y)\, dy + \lambda \int_a^b K_2(x, y) F(y)\, dy + \ldots \right\}. \tag{1.5}$$

The kernels $K_n(x, y)$ are related by the equations

$$K_n(x, y) = \int_a^b K(x, t) K_{n-1}(t, y)\, dt \quad (n = 2, 3, \ldots),$$

$$K_1(x, y) = K(x, y). \tag{1.6}$$

The same condition (1.4) implies that the following series converges:

$$K_1(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + \ldots \tag{1.7}$$

It is therefore permissible to interchange the order of integration and summation in representation (1.5). We introduce the notation $\Gamma(x, y, \lambda)$ for series (1.7); the function $\Gamma(x, y, \lambda)$ is called the *resolvent of Eq.* (1.1). The unknown representation of the solution of Eq. (1.1) is of the form

$$\varphi(x) = F(x) + \lambda \int_a^b \Gamma(x, y, \lambda) F(y)\, dy. \tag{1.8}$$

Thus, knowing the resolvent, we at once obtain the solution of the original equation with an arbitrary right-hand side (for a sufficiently small λ).

Note that inequality (1.4) is required for the convergence of the resulting series; the construction of the coefficients themselves only demands the integrability of the kernel $K(x, y)$.

If all coefficients $K_n(x, y)$ are now expressed in terms of the kernel $K(x, y)$ of the original equation, it is easy to obtain the following equalities:

$$K_n(x, y) = \underbrace{\int_a^b \int_a^b \ldots \int_a^b}_{n-1} K(x, t_{n-1}) K(t_{n-1}, t_{n-2}) \ldots$$

$$\ldots K(t_1, y)\, dt_1\, dt_2 \ldots dt_{n-1}, \tag{1.9}$$

$$K_{p+q}(x, y) = \int_a^b K_p(x, t) K_q(t, y)\, dt. \tag{1.10}$$

In the special case of $p = n - 1$, $q = 1$ we obtain

$$K_n(x, y) = \int_a^b K_{n-1}(x, t) K(t, y)\, dt \quad (n = 2, 3, \ldots). \tag{1.11}$$

We return to the representation of resolvent (1.7) and, assuming λ to be sufficiently small, consider a chain of equalities:

$$\Gamma(x, y, \lambda) = K_1(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + \ldots =$$

$$= K(x, y) + \lambda \int_a^b K(x, t) K_1(t, y)\, dt + \lambda \int_a^b K(x, t) K_2(t, y)\, dt + \ldots =$$

$$= K(x, y) + \lambda \int_a^b K(x, t) [K_1(t, y) + \lambda K_2(t, y) + \ldots]\, dt =$$

$$= K(x, y) + \lambda \int_a^b K(x, t) \Gamma(t, y, \lambda)\, dt. \qquad (1.12)$$

This relation may be regarded as a functional equation for the resolvent. Proceeding from formula (1.11), we obtain a different functional equation for the resolvent:

$$\Gamma(x, y, \lambda) = K(x, y) + \lambda \int_a^b K(t, y) \Gamma(x, t, \lambda)\, dt. \qquad (1.13)$$

Note that if condition (1.4) is fulfilled, the resolvent is an analytic function of the parameter λ in the circle $|\lambda| < \frac{1}{A(b-a)}$. It is under this restriction on the parameter λ that the resolvent has been defined in the above discussion. Relations (1.12) and (1.13) allow the resolvent to be determined in the whole plane of the complex variable, with the exception of some values.

Suppose that a function $\Gamma(x, y, \lambda)$ exists in the square $a \leqslant x \leqslant b$, $a \leqslant y \leqslant b$, prescribed for a certain value of λ and satisfying relations (1.12) and (1.13). We shall show that Eq. (1.1) has a solution representable in the form of (1.8). We multiply both sides of Eq. (1.1) by $\lambda\Gamma(y, x, \lambda)$ and integrate with respect to x. By transforming the resulting expression with the use of relation (1.13), we arrive at the equality

$$\lambda \int_a^b F(x) \Gamma(y, x, \lambda)\, dx - \lambda \int_a^b K(x, y) \varphi(y)\, dy = 0,$$

which, with (1.1), leads to the required representation:

$$\varphi(y) = \lambda \int_a^b \Gamma(y, x, \lambda) F(x)\, dx + F(y).$$

It remains to show that a function representable by expression (1.8) is a solution of Eq. (1.1). Indeed, by substituting (1.8) in

Eq. (1.1), we arrive at an identity [naturally, taking into account relation (1.12)].

We further prove that the resolvent is the ratio of two integral functions, analytic in the whole plane of the complex variable λ. We introduce into consideration the determinant

$$K\begin{pmatrix} x_1, & x_2, & \dots, & x_n \\ y_1, & y_2, & \dots, & y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_n) \\ K(x_2, y_1) & K(x_2, y_2) & \dots & K(x_2, y_n) \\ \dots & \dots & \dots & \dots \\ K(x_n, y_1) & K(x_n, y_2) & \dots & K(x_n, y_n) \end{vmatrix}. \quad (1.14)$$

By Hadamard's theorem (see E. Goursat [1]),* we obtain the following estimate:

$$K\begin{pmatrix} x_1, & x_2, & \dots, & x_n \\ y_1, & y_2, & \dots, & y_n \end{pmatrix} < n^{n/2}A^n, \quad (1.15)$$

since the sum of the squares of the elements of each row is less than nA^2.

We form the function

$$D(\lambda) = 1 - \frac{\lambda}{1!}\int_a^b K\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} dt_1 + \frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix} t_1, & t_2 \\ t_1, & t_2 \end{pmatrix} dt_1\, dt_2 - \dots$$

$$\dots + (-1)^n \frac{\lambda^n}{n!} \underbrace{\int_a^b \dots \int_a^b}_{n} K\begin{pmatrix} t_1, & t_2, & \dots, & t_n \\ t_1, & t_2, & \dots, & t_n \end{pmatrix} dt_1\, dt_2 \dots dt_n + \dots =$$

$$= 1 - \frac{\lambda}{1!} c_1 + \frac{\lambda^2}{2!} c_2 - \dots + (-1)^n \frac{\lambda^n}{n!} c_n + \dots \quad (1.16)$$

By using inequality (1.15), we obtain an estimate of each term of series (1.16) from which it follows that this series converges for all values of λ, and hence is an integral function called the *Fredholm determinant for the kernel* $K(x, y)$.

We introduce a new function $D\left(\begin{matrix} x \\ y \end{matrix} \middle| \lambda\right)$ assuming

$$\Gamma(x, y, \lambda) = \frac{D\left(\begin{matrix} x \\ y \end{matrix} \middle| \lambda\right)}{D(\lambda)}. \quad (1.17)$$

* If in an nth order determinant the sums of the squares of the elements of its rows are given, the determinant itself is less than the square root of the product of these sums.

From relations (1.12) we obtain an equation for determining the function $D\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle|\lambda\right)$:

$$D\left(\begin{matrix} x \\ y \end{matrix}\middle|\lambda\right)=\lambda\int_a^b K(x, t)\, D\left(\begin{matrix} t \\ y \end{matrix}\middle|\lambda\right) dt+D(\lambda) K(x, y). \quad (1.18)$$

We seek the solution of this equation in the form of a power series in λ by writing, for convenience, the function $D\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\middle|\lambda\right)$ as

$$D\left(\begin{matrix} x \\ y \end{matrix}\middle|\lambda\right)=q_0(x, y)-\frac{\lambda}{1!} q_1(x, y)-\ldots+ \\ +(-1)^n \frac{\lambda^n}{n!} q_n(x, y)+\ldots \quad (1.19)$$

We next substitute series (1.16) and (1.17) [with (1.19)] in Eq. (1.18), and equate the coefficients of like powers of λ:

$$q_0(x, y) = K(x, y), \quad (1.20)$$

$$q_n(x, y)=c_n K(x, y)-(-1)^{n-1} n \int_a^b K(x, t)\, q_{n-1}(t, y)\, dt. \quad (1.21)$$

These equalities enable us to calculate all coefficients $q_n(x, y)$ in succession, and obtain a general expression for them by introducing into consideration the function $L_n(x, y)$:

$$L_n(x, y)=\underbrace{\int_a^b \int_a^b}_{n} K\left(\begin{matrix} x, & t_1, & \ldots, & t_n \\ y, & t_1, & \ldots, & t_n \end{matrix}\right) dt_1\, dt_2 \ldots dt_n \quad (n=1, 2, \ldots). \quad (1.22)$$

It is obvious that $L_0(x, y) = K(x, y)$. It can also be shown by direct calculation that $L_1(x, y) = q_1(x, y)$; we shall prove that, in general, the following law holds:

$$L_n(x, y) = q_n(x, y) \quad (n = 2, 3, \ldots). \quad (1.23)$$

We shall first show that the functions $L_n(x, y)$ satisfy a relation coincident with (1.21):

$$L_n(x, y)=c_n K(x, y)-(-1)^{n-1} n \int_a^b K(x, t)\, L_{n-1}(t, y)\, dt. \quad (1.24)$$

From this and from the equality of the functions L_0 and q_0, L_1 and q_1 we obtain the general law (1.23).

Note that if two letters x_i or two letters y_i are transposed, the symbol $K\left(\begin{smallmatrix} x_1, & x_2, & \ldots, & x_n \\ y_1, & y_2, & \ldots, & y_n \end{smallmatrix}\right)$ only changes sign. We expand the de-

terminant $K\begin{pmatrix} x, & t_1, & \ldots, & t_n \\ y, & t_1, & \ldots, & t_n \end{pmatrix}$ by the elements of the first row remembering the above remark:

$$K\begin{pmatrix} x, & t_1, & \ldots, & t_n \\ y, & t_1, & \ldots, & t_n \end{pmatrix} = K(x, y)\, K\begin{pmatrix} t_1, & t_2, & \ldots, & t_n \\ t_1, & t_2, & \ldots, & t_n \end{pmatrix} -$$

$$-K(x, t_1)\, K\begin{pmatrix} t_1, & t_2, & \ldots, & t_n \\ y, & t_2, & \ldots, & t_n \end{pmatrix} - \ldots - K(x, t_n)\, K\begin{pmatrix} t_1, & t_2, & \ldots, & t_n \\ t_1, & t_2, & \ldots, & y \end{pmatrix}.$$

By integrating both sides of this relation with respect to all variables t_i, and interchanging the appropriate elements in each term,* we arrive at the required relation (1.24).

Referring again to Hadamard's theorem, we obtain the estimate

$$|q_n(x, y)| < (n+1)^{\frac{n+1}{2}} M^{n+1} (b-a)^n,$$

from which it follows that series (1.19) is an integral function.

Thus, it is proved that the resolvent is a meromorphic function of a complex variable. Since the resolvent $\Gamma(x, y, \lambda)$ exists for sufficiently small λ [inequality (1.4)] and its meromorphicity in the whole plane λ has been proved above, we conclude from the generalized Liouville theorem that the resolvent exists for all λ [except the values for which $D(\lambda) = 0$]. Consequently, the Fredholm equation (1.1) is uniquely solvable for any λ different from the zeros of the determinant $D(\lambda)$.

Let us now direct our attention to the values of λ ($\lambda = \lambda_0$) that make the Fredholm determinant zero and are called the *eigenvalues of an integral equation*. Since the determinant $D(\lambda)$ is an integral function, there can be only a finite number of eigenvalues in a limited part of the plane λ. It may happen that the same number λ_0 makes the function $D\left(\begin{matrix} x \\ y \end{matrix}\middle|\, \lambda\right)$ zero (for all values of x and y). Let us show that the multiplicity of the root in the numerator is less than it is in the denominator. Assume the representations

$$\begin{aligned} &D(\lambda) = (\lambda - \lambda_0)^k D_0(\lambda), \quad D_0(\lambda_0) \neq 0, \\ &D\left(\begin{matrix} x \\ y \end{matrix}\middle|\, \lambda\right) = (\lambda - \lambda_0)^l D_0\left(\begin{matrix} x \\ y \end{matrix}\middle|\, \lambda\right), \end{aligned} \tag{1.25}$$

where $D_0\left(\begin{matrix} x \\ y \end{matrix}\middle|\, \lambda\right)$ is a series in terms of positive powers of λ, whose free term is different from zero for some values of x and y. It is obvious that $D'(\lambda)$ has a zero of order $k - 1$ at the point λ_0.

* In the ith term t_i and t_1 are interchanged.

We set $y = x$ on both sides of formula (1.19) and integrate with respect to x:

$$\int_a^b D\left(\begin{matrix} x \\ x \end{matrix}\middle|\, \lambda\right) dx = \int_a^b K(x, x)\, dx + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_a^b L_n(x, x)\, dx =$$

$$= c_1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n+1} K\begin{pmatrix} x, & t_1, & t_2, & \dots, & t_n \\ x, & t_1, & t_2, & \dots, & t_n \end{pmatrix} dt_1 \dots$$

$$\dots dt_n\, dx = c_1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} c_{n+1} = -D'(\lambda). \quad (1.26)$$

For the case under consideration this equality may be rewritten in an alternate form:

$$D'(\lambda) = -(\lambda - \lambda_0)^l \int_a^b D_0\left(\begin{matrix} x \\ x \end{matrix}\middle|\, \lambda\right) dx. \quad (1.27)$$

It may happen that the integration will yield one more power of the factor $(\lambda - \lambda_0)$, from which it follows that $k - 1 \geqslant l$ and $k > l$. Thus, the poles of the resolvent must necessarily coincide with the zeros of the Fredholm determinant.

Suppose that λ_0 is a pole of multiplicity r of the resolvent. We then have the expansion*

$$\frac{D\left(\begin{matrix} x \\ y \end{matrix}\middle|\, \lambda\right)}{D(\lambda)} = \frac{a_{-r}(x, y)}{(\lambda - \lambda_0)^r} + \frac{a_{-r+1}(x, y)}{(\lambda - \lambda_0)^{r-1}} + \dots + \frac{a_{-1}(x, y)}{(\lambda - \lambda_0)} +$$

$$+ \sum_{i=0}^{\infty} a_i(x, y)(\lambda - \lambda_0)^i. \quad (1.28)$$

Substituting series (1.28) in the functional equation (1.12), multiplying it successively by the factors $(\lambda - \lambda_0)^n$ $(n = r, r-1, \dots)$, and then setting $\lambda = \lambda_0$, we obtain the relations

$$a_{-r}(x, y) = \lambda_0 \int_a^b K(x, t)\, a_{-r}(t, y)\, dt, \quad (1.29)$$

$$a_{-r+1}(x, y) - \frac{a_{-r}(x, y)}{\lambda_0} = \lambda_0 \int_a^b K(x, t)\, a_{-r+1}(t, y)\, dt, \quad (1.30)$$

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* The parameter λ in the coefficients $a_n(x, y)$ is omitted.

etc. It follows from relation (1.29) that the coefficient $a_{-r}(x, y)$ as a function of x with an arbitrary fixed value of y (regarded as a parameter) is the solution of the homogeneous equation

$$\varphi(x) = \lambda_0 \int_a^b K(x, y)\, \varphi(y)\, dy.$$

The non-trivial solutions of a homogeneous equation are called the *eigenfunctions* (or *zeros*) *corresponding to the eigenvalue* λ_0.

Repeating the above reasoning with reference to Eq. (1.13), we obtain relations similar to (1.29) and (1.30). In particular, the coefficient $a_{-r}(x, y)$ as a function of y with a fixed value of x is the eigenfunction of the homogeneous equation

$$\psi(x) = \lambda_0 \int_a^b K(y, x)\, \psi(y)\, dy, \tag{1.31}$$

called the *companion* (or *transposed*) *equation to* (1.1)

To construct the complete theory, it is necessary to study the question of the solvability of integral equations on the eigenvalues. It is obvious that the resolvent of the companion equation is obtained from the resolvent of the original equation by transposing the variables. Consequently, the Fredholm determinants of the original and companion equations are identical, and so are the eigenvalues.

Let us prove that the number of eigenfunctions corresponding to the same value λ_0 is finite (implying linearly independent solutions). Let $\varphi_1^*(x), \varphi_2^*(x), \ldots, \varphi_m^*(x)$ be orthonormal eigenfunctions corresponding to the number λ_0. Consider the equalities that are satisfied by these functions:

$$\frac{\varphi_i^*(x)}{\lambda_0} = \int_a^b K(x, y)\, \varphi_i^*(y)\, dy. \tag{1.32}$$

It is obvious that the right-hand side is the Fourier coefficient of the function $K(x, y)$ (as a function of the argument y) in the orthonormal system of functions $\varphi_i^*(y)$. From Bessel's inequality (m is the number of eigenfunctions) it follows that

$$\sum_{i=1}^{m} \frac{\varphi_i^{*2}(x)}{\lambda_0^2} \leqslant \int_a^b K^2(x, y)\, dy.$$

Remembering that the eigenfunctions are normalized, and integrating both sides of the last inequality with respect to x, we obtain

$$\frac{m}{\lambda_0^2} \leqslant \int_a^b \int_a^b K^2(x, y)\, dx\, dy < \infty.$$

It follows from this estimate that the number of eigenfunctions is finite.

Let us prove that the numbers of eigenfunctions of the original and companion equations (of course, for the same eigenvalue) are equal. Suppose that there are m orthonormal eigenfunctions $\varphi_j^*(x)$ of the original equation and n functions of the companion equation denoted by $\psi_j^*(x)$. Assume that $m < n$, and consider two companion equations

$$\varphi(x) = \lambda_0 \int_a^b \left[K(x, y) - \sum_{j=1}^m \psi_j^*(x)\, \varphi_j^*(y) \right] \varphi(y)\, dy, \qquad (1.33)$$

$$\psi(x) = \lambda_0 \int_a^b \left[K(y, x) - \sum_{j=1}^m \psi_j^*(y)\, \varphi_j^*(x) \right] \psi(y)\, dy. \qquad (1.34)$$

Let us prove that Eq. (1.33) has no eigenfunctions. Multiply this equation by any one of the functions $\psi_j^*(x)$ $(j \leqslant m)$ and integrate with respect to x. By interchanging the order of integration in the double integral [keeping in mind the orthonormality of the functions $\psi_j^*(x)$] we arrive at the equality

$$\int_a^b \varphi_j^*(x)\, \varphi(x)\, dx = 0, \qquad (1.35)$$

which is fulfilled for all j if $j \leqslant m$. Consequently, every solution of Eq. (1.33) satisfies Eq. (1.1) with the zero right-hand side, i.e., it is an eigenfunction. This solution must therefore be represented as a sum:

$$\varphi(x) = \sum_{j=1}^m c_j \varphi_j^*(x).$$

Multiply both sides of this equality by any function $\varphi_k^*(x)$ and integrate with respect to x:

$$\int_a^b \varphi(x)\, \varphi_k^*(x)\, dx = \sum_{j=1}^m c_j \int_a^b \varphi_k^*(x)\, \varphi_j^*(x)\, dx.$$

It follows that $c_k = 0$, and hence Eq. (1.33) has no eigenfunctions. Direct substitution proves that the functions $\psi_j^*(x)$ $(j > m)$ are the solutions of Eq. (1.34). Equations (1.33) and (1.34) are companion and, as proved above, their eigenvalues must coincide. But Eq. (1.33) does not have λ_0 as an eigenvalue, whereas (1.34) does. We thus come to a contradiction.

We now turn to a direct investigation of the solvability of the integral equation (1.1). Consider the equation

$$\varphi(x)=\lambda_0 \int_a^b \Big[K(x, y)-\sum_{j=1}^{m} \psi_j^*(x)\,\varphi_j^*(y)\Big]\varphi(y)\,dy+F(x), \qquad (1.36)$$

which, as follows from the above discussion, is solvable with any right-hand side. Proceeding as before, i.e., multiplying both sides by any function $\psi_k^*(x)$, and integrating, we arrive at the equality

$$\lambda_0 \int_a^b \psi_k^*(x)\,\varphi(x)\,dx=\int_a^b \psi_k^*(x)\,F(x)\,dx. \qquad (1.37)$$

If we require the fulfilment of the conditions

$$\int_a^b \psi_k^*(x)\,F(x)\,dx=0, \qquad (1.38)$$

Eq. (1.36) merely transforms into the original equation (1.1) (since there are m such equalities), which is thus solvable. We shall also prove that conditions (1.38) are necessary conditions for the solvability of the original equation (1.1).To do this, we multiply (1.1) by any function $\psi_k^*(x)$ and integrate with respect to x. By interchanging the order of integration in the double integral, we obtain, after appropriate cancellation, the same relations (1.38).

Thus, conditions (1.38) are necessary and sufficient conditions for the existence of the solution of an integral equation on the eigenvalues. The solution of such equations cannot be unique; it is represented to within a sum of the form

$$\sum_{k=1}^{m} c_k \varphi_k^*(x).$$

In conclusion we shall prove inequalities which will be needed in what follows. Let λ_0 be an eigenvalue and let, naturally, conditions (1.38) be fulfilled. Then the following equalities hold:

$$\int_a^b \varphi_k(x)\,\psi_j^*(x)\,dx=0 \qquad \begin{pmatrix} j=1, 2, \ldots, n \\ k=1, 2, \ldots \end{pmatrix}.$$

here $\varphi_k(x)$ are the terms of series (1.2).

Let us show the validity of these equalities for the function $\varphi_1(x)$:

$$\int_a^b \varphi_1(x)\,\psi_1^*(x)\,dx=\lambda \int_a^b \int_a^b F(y)\,K(x, y)\,\psi_1^*(x)\,dy\,dx=$$

$$=\int_a^b F(y)\,\psi_1^*(y)\,dy.$$

The transition to the other functions $\varphi_k(x)$ is obvious and is made by induction.

Note that Eqs. (1.29), (1.30) and similar ones make it possible to establish the relation between the coefficients $a_{-r}(x, y)$ (for negative values of the subscript) in the expansion of the resolvent in the neighbourhood of a pole and the corresponding eigenfunctions of the original and companion equations. Below is the final expression for the irregular part:

$$\frac{\lambda_0\lambda^{r-1}\varphi_1(x)\psi_r(y)}{(\lambda-\lambda_0)^r}+\dots$$

$$\dots\frac{\lambda_0\lambda[\varphi_1(x)\psi_2(y)+\varphi_2(x)\psi_3(y)+\dots+\varphi_{r-1}(x)\psi_r(y)]}{(\lambda-\lambda_0)^2}+$$

$$+\frac{\lambda_0[\varphi_1(x)\psi_1(y)+\dots+\varphi_r(x)\psi_r(y)]}{\lambda-\lambda_0}, \qquad (1.39)$$

where $\varphi_j(x)$ and $\psi_j(x)$ are the systems of eigenfunctions related by the equalities

$$\lambda_0\int_a^b \varphi_j(x)\varphi_k(x)\,dx=\delta_{jk}, \quad \lambda_0\int_a^b \psi_j(x)\psi_k(x)\,dx=\delta_{jk},$$

$$\int_a^b \varphi_j(x)\psi_k(x)\,dx=0 \quad (k\neq j-1,\ j),$$

$$\lambda_0\int_a^b \varphi_j(x)\psi_k(x)\,dx=1, \quad (k=j-1,\ j).$$

This expression shows in a vivid way the meaning of the foregoing conditions (1.38) for the solvability of an integral equation on the eigenvalue. Indeed, we refer to the representation of the solution by means of resolvent (1.8). It follows from the orthogonality conditions that the solution is an analytic function of the parameter λ. In the case when λ_0 is the numerically smallest pole of the resolvent, the solution can therefore be obtained directly by the method of successive approximations.

It was assumed above that the kernel of the integral equation was bounded, and hence by using rather crude estimates we determined a sufficiently small value of λ for which the representation of the solution in the form of a series was convergent, and this was a prerequisite in the further development of the whole theory. It appears to be possible, however, to extend the theory of the resolvent to integral equations with less severe restrictions on the kernels. Suppose that a kernel has a weak singularity. We transform Eq. (1.1) by replacing the function $\varphi(y)$ on the right-hand side by its integral

representation following from the equation itself:

$$\varphi(x) = F(x) + \lambda^2 \int_a^b K_2(x, y)\varphi(y)\,dy + \lambda \int_a^b K(x, y) F(y)\,dy. \quad (1.40)$$

Since $F(x)$ and $\int_a^b K(x, y) F(y)\,dy$ are the first terms in the series expansion of the solution (1.2), we conclude that the unknown function $\varphi(x)$ also satisfies the equation

$$\varphi(x) = \lambda^2 \int_a^b K_2(x, y)\varphi(y)\,dy + \varphi_0(x) = \lambda\varphi_1(x).$$

It is not difficult to show that the following general formula for any integer n holds true:

$$\varphi(x) = \lambda^n \int_a^b K_n(x, y)\varphi(y)\,dy + S_n(x), \quad (1.41)$$

$$S_n(x) = \varphi_0(x) + \lambda\varphi_1(x) + \ldots + \lambda^{n-1}\varphi_{n-1}(x).$$

Let the number n be such that the kernel $K_n(x, y)$ is bounded. In the region of sufficiently small λ, we can construct the resolvent of this equation, which will be denoted by $\Gamma_n(x, y, \lambda^n)$. It is obvious that every solution of Eq. (1.1) satisfies Eq. (1.41). We shall also prove the converse, i.e., that the solution of Eq. (1.41) satisfies Eq. (1.1). Assume

$$\omega(x) = \varphi(x) - \lambda \int_a^b K(x, y)\varphi(y)\,dy - F(x), \quad (1.42)$$

where $\varphi(x)$ is the solution of Eq. (1.41).

Let us show that this function satisfies the homogeneous integral equation (1.41) and hence is zero (since λ^n is assumed to be different from the eigenvalue). We first obtain the required representation for the integral:

$$\lambda \int_a^b K(x, y) S_n(y)\,dy = \lambda \int_a^b K(x, y)\varphi_0(y)\,dy +$$

$$+ \ldots + \lambda^n \int_a^b K(x, y)\varphi_{n-1}(y)\,dy = S_{n+1}(x) - \varphi_0(x) =$$

$$= S_n(x) + \lambda^n \int_a^b K(x, y) F(y)\,dy - F(x). \quad (1.43)$$

Further we transform (1.42) substituting equality (1.41) in the integral term and using (1.43). Below are given the necessary transformations:

$$\omega(x)=\varphi(x)-\lambda\int_a^b K(x,y)\varphi(y)\,dy-F(x)=$$

$$=\varphi(x)-F(x)-\lambda\int_a^b K(x,y)\left\{\lambda^n\int_a^b K_n(y,t)\varphi(t)\,dt+S_n(y)\right\}dy=$$

$$=\varphi(x)-S_n(x)-\lambda^n\int_a^b K_n(x,y)F(y)\,dy-$$

$$-\lambda^{n+1}\int_a^b \varphi(t)\int_a^b K_n(y,t)K(x,y)\,dy\,dt=$$

$$=\lambda^n\int_a^b K_n(x,y)\varphi(y)\,dy-\lambda^n\int_a^b K_n(x,y)F(y)\,dy-$$

$$-\lambda^{n+1}\int_a^b \varphi(t)\int_a^b K_n(x,y)K(y,t)\,dy\,dt=$$

$$=\lambda^n\int_a^b K_n(x,y)\left\{\varphi(y)-F(y)-\lambda\int_a^b K(y,t)\varphi(t)\,dt\right\}dy=$$

$$=\lambda^n\int_a^b K_n(x,y)\,\omega(y)\,dy. \quad (1.44)$$

In the last equality use has been made of the relation following from (1.10):

$$\int_a^b K_m(x,y)K_n(y,t)\,dy=\int_a^b K_m(y,t)K_n(x,y)\,dy.$$

Let us establish the relation between the resolvent of Eqs. (1.1) and (1.41). Let $\Gamma_n(x, y, \lambda^n)$ be the resolvent of the iterated kernel $K_n(x, y)$:

$$\Gamma_n(x,y,\lambda^n)=K_n(x,y)+\lambda K_{2n}(x,y)+\ldots+\lambda^{p-1}K_{pn}(x,y)+\ldots \quad (1.45)$$

It is clear that this function is expressible in terms of $\Gamma(x, y, \lambda)$. Of greater interest to us is, on the contrary, the representation of the resolvent $\Gamma(x, y, \lambda)$ in terms of $\Gamma_n(x, y, \lambda^n)$. We have the obvious

equality

$$\int_a^b K_i(x, t)\,\Gamma_n(t, y, \lambda^n)\,dt = K_{n+i}(x, y) + \lambda K_{2n+i}(x, y) + \dots \qquad (1.46)$$

By rearranging the terms in formula (1.12), we obtain

$$\Gamma(x, y, \lambda) = K(x, y) + \lambda K_2(x, y) + \dots + \lambda^{n-2} K_{n-1}(x, y) +$$
$$+ \lambda^{n-1}\Gamma_n(x, y, \lambda^n) + \lambda^n \int_a^b K(x, t)\,\Gamma_n(t, y, \lambda^n)\,dt +$$
$$+ \dots + \lambda^{2n-2} \int_a^b K_{n-1}(x, t)\,\Gamma_n(t, y, \lambda^n)\,dt.$$

It follows from this equality that in a sufficiently small neighbourhood of λ the representation of the resolvent is a convergent series. The meromorphicity of the resolvent then implies its existence everywhere except for the zeros of the Fredholm determinant.

The foregoing theory can also be applied to the case when the kernels are of the form

$$K(x, y) = \frac{H(x, y)}{(x-c)^{\alpha}},$$

where the function $H(x, y)$ is bounded and $\alpha < 1$. In contrast to the case studied above, it is impossible to obtain a bounded kernel by a finite number of iterations.

Consider a sequence of function $H^{(n)}(x, y)$ obtained by the recurrence formula

$$H^{(n)}(x, y) = \int_a^b \frac{H(x, t)\,H^{(n-1)}(t, y)}{|t-c|^{\alpha}}\,dt$$

$$(n = 1, 2, \dots), \quad H^1(x, y) = H(x, y).$$

It is not difficult to show that $|H^{(n)}| < M^h h^{n-1}$, where $h = \frac{(c-a)^{1-\alpha} + (b-c)^{1-\alpha}}{1-\alpha}$ and $M = \max |H(x, y)|$.

We define the function $\Gamma(x, y, \lambda)$ by the expression

$$\Gamma(x, y, \lambda) = \frac{1}{|x-c|^{\alpha}} \{H(x, y) - \lambda H^{(2)}(x, y) + \dots\}. \qquad (1.47)$$

The numerator on the right-hand side converges uniformly for a sufficiently small λ ($|\lambda| < 1/Mh$). By direct substitution we verify that the function defined by (1.47) satisfies Eqs. (1.12) and (1.13) and hence is the resolvent.

The above results are obtained by studying directly the integral equation. It is sometimes advisable, however, to draw on the original differential equations to investigate the properties of the resolvent (if the integral equation arose from the solution of a boundary value problem). For example, this combination made it possible to carry out a comprehensive analysis of the integral equations of potential theory (see N. M. Günter [1]). In Secs. 29, 31 it is in this way that we study the integral equations for the three-dimensional problem of the theory of elasticity.

2. Cauchy-type Integral

Denote by L a smooth closed contour in the plane of the complex variable z. The inner region bounded by the contour L is denoted by D^+, and the outer region by D^-. Let $f(\tau)$ be the boundary value on the contour L of a function analytic inside or outside the contour. We first assume that the function $f(\tau)$ is continuous. It follows from Cauchy's integral theorem that in the first case the integral

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau-z}\,d\tau, \tag{2.1}$$

called *Cauchy's integral*, for $z \notin L$ is equal to the above analytic function when $z \in D^+$ and vanishes when $z \in D^-$. In the second case Cauchy's integral is zero when $z \in D^+$ and restores the function when $z \in D^-$ with opposite sign (if it vanishes at infinity).*

We further reject the assumption that the continuous function (density function of the integral) is the boundary value (from the inside or outside) of an analytic function. In this case an integral of the form of (2.1) is called a *Cauchy-type integral.* Let us show that the Cauchy-type integral represents functions analytic in the region D^+ and D^- and denoted by $\Phi^+(z)$ and $\Phi^-(z)$, respectively. The functions thus constructed may be designated by a single piecewise analytic function $\Phi(z)$.

We form the difference $\Phi(z+\Delta z)-\Phi(z)$, the points $z+\Delta z$ and z both belonging to D^+ or D^-. Consider the difference

$$\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z}-\int_L \frac{f(\tau)}{(\tau-z)^2}\,d\tau =$$

$$=\int_L \left[\left(\frac{1}{\tau-z-\Delta z}-\frac{1}{\tau-z}\right)\cdot\frac{f(\tau)}{\Delta z}-\frac{f(\tau)}{(\tau-z)^2}\right]d\tau.$$

Since the kernel $\frac{1}{\tau-z}$ is analytic in z (when $z\notin L$), the bracketed expression can be made arbitrarily small if Δz is sufficiently small. By applying a limiting process, we ascertain the existence of the

* The sense of description is chosen counterclockwise throughout.

limit of $[\Phi(z+\Delta z)-\Phi(z)]/\Delta z$, equal to the derivative of the function. From the differentiability of a function of a complex variable it follows, as is known, that the function is analytic.

Cauchy-type integrals can be studied assuming the functions $f(\tau)$ to belong to one class or another. We shall restrict our study to the case when the density functions of Cauchy-type integrals belong to the Hölder-Lipschitz class (the class H-L). Their modulus of continuity $\omega(\delta)$ is a power-law function:

$$\omega(\delta)=\sup|f(\tau_2)-f(\tau_1)|\leqslant A|\tau_2-\tau_1|^{\lambda},$$
$$\delta=|\tau_2-\tau_1| \quad (A>0,\ 0<\lambda\leqslant 1).$$

The choice of this class of functions is justified by the possibility of constructing a comprehensive mathematical theory in this case and also by the possibility of describing a number of applied problems of mathematical physics.

It is obvious that the sum, product and quotient (if the denominator is not zero) of two functions also belong to the class H-L with the smallest index. Note that differentiable functions belong to the class H-L with the index $\lambda = 1$.

The property of a function to belong to the class H-L is a local property. It may be fulfilled in the neighbourhood of one point of the contour and may not be fulfilled in the neighbourhood of another. In the following discussion we shall only consider functions that belong to the class H-L on the entire contour. The non-fulfilment of this condition at particular points will be specially noted. The belonging of a given function $\varphi(\tau)$ to the class H-L will be further designated as $\varphi(\tau)\in H(A,\lambda)$ or $\varphi(\tau)\in H$.

In defining the Cauchy-type integral it was assumed that the point $z\notin L$. Consider, now, the integral

$$\frac{1}{2\pi i}\int\limits_L \frac{\varphi(\tau)}{\tau-t}\,d\tau \quad (t\in L). \tag{2.2}$$

Integral (2.2) does not exist in the ordinary sense (as an improper integral) since the singularity of the integrand is unity.

Let us define integral (2.2) as follows. From the point t as a centre we draw a circle of sufficiently small radius δ. The value of δ is chosen from the condition that a circle of any radius $\rho<\delta$ centred at the point t should have only two points of intersection with the contour L. The points of intersection of the circle of radius δ and the curve L are denoted by t_1 and t_2 (in the counterclockwise sense of description) and L_δ denotes the minor arc joining the points t_1 and t_2 (Fig. 1). In the integral

$$\frac{1}{2\pi i}\int\limits_{L-L_\delta} \frac{\varphi(\tau)}{\tau-t}\,d\tau \tag{2.3}$$

the integrand is bounded and hence the integral itself exists for all values of δ $(\delta \neq 0)$. If a limiting process is applied by allowing δ to tend to zero, the limit thus obtained is called the *singular value of the Cauchy-type integral* or the *Cauchy principal value*. Note that the difference between the singular value of the integral and the

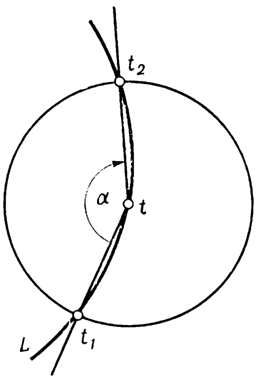

Fig. 1. Positions of points on the contour of integration

improper value is that in the first case the ratio of the arc lengths, (t_2, t) and (t_1, t), is not arbitrary, but must tend to unity in the limit.

Let us now direct our attention to the simplest singular integral $[\varphi(\tau) = 1]$:

$$\int_L \frac{d\tau}{\tau - t} = \lim_{\delta \to 0} \int_{L - L_\delta} \frac{d\tau}{\tau - t} = \lim_{\delta \to 0} [\ln(t_2 - t) - \ln(t_1 - t)].$$

By allowing δ to tend to zero, and taking into account the smoothness of the contour L, we obtain the final expression for the singular integral in question:

$$\frac{1}{2\pi i} \int_L \frac{d\tau}{\tau - t} = \frac{1}{2}. \tag{2.4}$$

We now turn to the general case. The singular integral (2.2) may be represented as

$$\frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \frac{\varphi(t)}{2\pi i} \int_L \frac{d\tau}{\tau - t} =$$

$$= \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \frac{\varphi(t)}{2}. \tag{2.5}$$

The first integral is improper since $\varphi(t) \in H(A, \lambda)$; the second integral has been considered above.

Consider, now, the limiting values of the functions $\Phi^+(z)$ and $\Phi^-(z)$ as the points z tend to the points t of the contour L (from the inside and outside, respectively) and denote these limiting values by $\Phi^\pm(t)$. Let us fix a point t on the contour L and consider the integral

$$\psi_t(z) = \int_L \frac{\varphi(\tau)-\varphi(t)}{\tau - z}\, d\tau. \tag{2.6}$$

Consider this function at points z situated on a line intersecting the contour L at the point t. Let us prove that there exist limiting

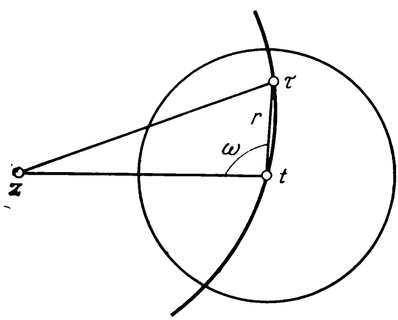

Fig. 2. Positions of points in a plane and on the contour of integration

values $\psi_t^+(t)$ and $\psi_t^-(t)$ and also a direct value $\psi_t(t)$. We form the following difference:

$$\psi_t(z) - \psi_t(t) = \int_L (z-t)\, \frac{\varphi(\tau)-\varphi(t)}{(\tau-z)(\tau-t)}\, d\tau.$$

The contour L is divided into parts $L - L_\delta$ and L_δ (the arc L_δ is defined as before) and accordingly the integral is represented as the sum of two terms: I_1, along the arc L_δ, and I_2, along the arc $L - L_\delta$. For simplicity, we restrict ourselves to the case when the points z approach along a non-tangential path. In this case the angle ω (Fig. 2) is always greater than a certain angle $\omega_0 > 0$, and the following inequality is obvious:

$$\left|\frac{z-t}{\tau - z}\right| \geqslant \frac{1}{\sin\omega_0} = K.$$

From the H-L condition it follows that

$$\left|\frac{\varphi(\tau)-\varphi(t)}{\tau - t}\right| < Ar^{\lambda-1}, \quad r = |\tau - t|.$$

Since the contour L is smooth, there exists a constant m greater than $\left|\frac{d\tau}{dr}\right|$. We apply the foregoing inequalities to estimate the

integral I_1:

$$|I_1| \leqslant \int_{L_\delta} \frac{|z-t|}{|\tau-z|} \left| \frac{\varphi(\tau)-\varphi(t)}{\tau-t} \right| |d\tau| <$$

$$< KAm \int_{L_\delta} r^{\lambda-1} |dr| = 2KAm \int_0^\delta r^{\lambda-1}\, dr = \frac{2KAm\delta^\lambda}{\lambda}.$$

By assigning a sufficiently small value of δ, we can make the integral I_1 less than any preassigned number ε. To estimate I_2, we choose z sufficiently close to the point t so that $|I_2|$ will be less than ε, which is possible because of the continuity of the integrand. We have the estimate

$$|I_1| + |I_2| < 2\varepsilon,$$

from which it follows that the function $\psi_t(z)$ is continuous.

Thus, it may be considered that the following equalities are proved:

$$\lim_{z\to t,\, z\in D^+} \psi_t^+(z) = \lim_{z\to t,\, z\in D^-} \psi_t^-(z) = \psi_t(t). \tag{2.7}$$

Here $\psi_t(t)$ is the direct value of integral (2.6), i.e., the value obtained by substituting $z = t$. Since the integral

$$\frac{1}{2\pi i} \int_L \frac{d\tau}{\tau - z} = \begin{cases} 1, & z \in D^+, \\ 0, & z \in D^-, \end{cases}$$

and, according to (2.4), its singular value is 1/2, it follows from equality (2.7) that

$$\Phi^+(t) - \varphi(t) = \Phi^-(t) = \psi_t(t) = \Phi(t) - \frac{1}{2}\varphi(t). \tag{2.8}$$

By eliminating the auxiliary continuous function $\psi(t)$ from these equalities, we arrive at the Sokhotskii-Plemelj formulas playing a very important part in the following discussion:

$$\begin{aligned} \Phi^+(t) &= \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau, \\ \Phi^-(t) &= -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau, \end{aligned} \tag{2.9}$$

or in an alternate form

$$\varphi(t) = \Phi^+(t) - \Phi^-(t), \quad \Phi(t) = \frac{1}{2}[\Phi^+(t) + \Phi^-(t)]. \tag{2.9'}$$

It follows directly from the Sokhotskii-Plemelj formulas that the limiting values $\Phi^\pm(t)$ are continuous functions. It appears, however, that these functions belong to the class H-L with the same

index λ if $\lambda < 1$ or with the index $1 - \varepsilon$ $(\varepsilon > 0)$ if $\lambda = 1$. This result is called the *Plemelj-Privalov theorem.*

Obviously, it is sufficient to prove this theorem for function (2.6). Let us estimate the difference for two points, t_1 and t_2, a sufficiently small distance apart:

$$\psi_{t_2}(t_2) - \psi_{t_1}(t_1) = \frac{1}{2\pi i} \int\limits_L \left\{ \frac{\varphi(\tau) - \varphi(t_2)}{\tau - t_2} - \frac{\varphi(\tau) - \varphi(t_1)}{\tau - t_1} \right\} d\tau. \quad (2.10)$$

Since the contour L is assumed to be smooth, we have the estimate $s(t_1, t_2) \leqslant m \mid t_2 - t_1 \mid$, where $s(t_1, t_2)$ is the length of the minor arc of the contour between the points t_1 and t_2. Let us isolate an arc L_1 on the contour L by laying off, on both sides of the point t_1, arcs whose lengths are equal to $2s(t_1, t_2)$. We have then

$$\psi_{t_2}(t_2) - \psi_{t_1}(t_1) = \frac{1}{2\pi i} \int\limits_{L_1} \frac{\varphi(\tau) - \varphi(t_2)}{\tau - t_2} d\tau - \frac{1}{2\pi i} \int\limits_{L_1} \frac{\varphi(\tau) - \varphi(t_1)}{\tau - t_1} d\tau +$$

$$+ \frac{1}{2\pi i} \int\limits_{L-L_1} \frac{\varphi(t_1) - \varphi(t_2)}{\tau - t_1} d\tau + \frac{1}{2\pi i} \int\limits_{L-L_1} \frac{[\varphi(\tau) - \varphi(t_2)](t_2 - t_1)}{(\tau - t_1)(\tau - t_2)} d\tau =$$

$$= I_1 + I_2 + I_2 + I_4.$$

We estimate the integral I_1:

$$|I_1| < \frac{1}{2\pi} \int\limits_{L_1} \left| \frac{\varphi(\tau) - \varphi(t_2)}{\tau - t_2} \right| |d\tau| \leqslant C_1 \int\limits_{L_1} \frac{|d\tau|}{|\tau - t_2|^{1-\lambda}} \leqslant$$

$$\leqslant C_2 \int\limits_0^{|t_2 - t_1|} \frac{ds}{s^{1-\lambda}} \leqslant C_3 s^{\lambda}(t_1, t_2) \leqslant A_1 |t_2 - t_1|^{\lambda}.$$

The meaning of the constants introduced is obvious. The estimate for the integral I_2 is constructed in a similar way. We estimate the integral I_3:

$$|I_3| \leqslant \frac{A|t_2 - t_1|^{\lambda}}{2\pi} \left| \int\limits_{L-L_1} \frac{d\tau}{\tau - t_1} \right|.$$

Since the last integral is bounded, we arrive at the required estimate:

$$|I_3| \leqslant A_3 |t_2 - t_1|^{\lambda}.$$

We form a chain of inequalities for the integral I_4:

$$|I_4| \leqslant A \frac{|t_2 - t_1|}{2\pi} \int\limits_{L-L_1} \frac{ds}{|\tau - t_1| \, |\tau - t_2|^{1-\lambda}} \leqslant$$

$$\leqslant A' |t_2 - t_1| \int\limits_{L-L_1} \frac{ds}{s(t_1, \tau) \cdot s^{1-\lambda}(t_2, \tau)} =$$

$$= A' |t_2 - t_1| \int\limits_{L-L_1} \frac{ds}{s^{2-\lambda}(t_1, \tau) \left[\frac{s(t_2, \tau)}{s(t_1, \tau)} \right]^{1-\lambda}}.$$

From the method of constructing the arc L_1 it follows that $s(t_2, \tau)/s(t_1, \tau) \geqslant 1/2$; hence,

$$|I_4| \leqslant A\ |t_2 - t_1| \int\limits_{L-L_1} \frac{ds}{s^{2-\lambda}(t_1, \tau)}.$$

Thus, if $\lambda < 1$, we have the inequality $|I_4| \leqslant A_4 |t_2 - t_1|^\lambda$. If $\lambda = 1$, then $|I_4| \leqslant A_4' |t_2 - t_1| \cdot |\ln |t_2 - t_1||$. This inequality can be weakened:* $|I_4| \leqslant A_4' |t_2 - t_1|^{1-\varepsilon}$ $(\varepsilon > 0)$.

The Plemelj-Privalov theorem follows from all the estimates obtained above.

Consider, further, the case when the Cauchy-type integral is taken along an unbounded curve. By way of example we choose the real axis. We require that the density function $\varphi(t) \in H(A, \lambda)$ at all interior points, and that the following condition should be fulfilled at infinity:

$$|\varphi(t) - \varphi(\infty)| < \frac{A}{|\tau|^\mu} \quad (\mu > 0). \tag{2.11}$$

If $\varphi(\infty) \neq 0$, the question arises as to the existence of the Cauchy-type integral

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi(\tau)}{\tau - z} d\tau, \tag{2.12}$$

since it does not exist as an improper integral because of the unboundedness of the limits of integration. We agree to understand by integral (2.12) the following limit:

$$\lim_{N \to \infty} \frac{1}{2\pi i} \int\limits_{-N}^{N} \frac{\varphi(\tau)}{\tau - z} d\tau.$$

Below is a regular representation for integral (2.12):

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi(\tau)\, d\tau}{\tau - z} = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi(\tau) - \varphi(\infty)}{\tau - z} d\tau \pm \frac{1}{2} \varphi(\infty),$$

where the plus sign is taken when $\operatorname{Im} z > 0$, and the minus sign when $\operatorname{Im} z < 0$.

The foregoing results pertaining to Cauchy-type integrals and singular integrals for closed contours are completely extended to the case of unbounded contours subject to the restrictions specified above.

Let us now examine the question of interchanging the order of integration in iterated integrals; we first consider the case when one

* To obtain an estimate in the class of functions satisfying the H-L condition.

integral is ordinary:

$$I(z)=\int_L \omega(\tau,\ z)\,d\tau \int_L \frac{\varphi(\tau,\ \tau_1)}{\tau_1-\tau}\,d\tau_1. \tag{2.13}$$

Let the function $\varphi(\tau,\ \tau_1)\in H(A,\ \lambda)$ with respect to each variable, and let the function $\omega(\tau,\ z)$ be integrable with respect to τ for values of z from a certain set. Consider an integral $I_1(z)$ obtained from (2.13) by interchanging the order of integration:

$$I_1(z)=\int_L d\tau_1 \int_L \frac{\omega(\tau,\ z)\,\varphi(\tau,\ \tau_1)}{\tau_1-\tau}\,d\tau. \tag{2.14}$$

For clarity, we shall define the position of the points τ and τ_1 on the arc by arc abscissas s and s_1 measured from some fixed point

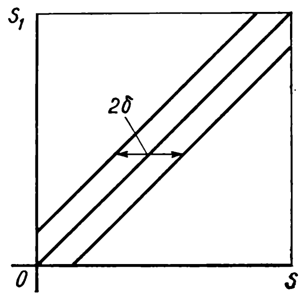

Fig. 3. Region of integration in a double integral

(Fig. 3). We cut out from the region of integration (square of side l) a strip whose mean line coincides with a diagonal, and the width (in the direction of either side of the square) is δ (δ is a small quantity). Denote the strip by L_δ.

Represent each of the integrals, (2.13) and (2.14), in the form

$$I=I_0+I_\delta,\quad I_1=I_{10}+I_{1\delta},$$

where $I_0=I_{10}$; because of the possibility of interchanging in regular integrals the subscript δ indicates that the region of integration is L_δ, and the subscript zero denotes the remainder. We further have

$$I_\delta=\int_L \omega(\tau,\ z)\,d\tau \int_{l_{\delta(\tau)}} \frac{\varphi(\tau,\ \tau_1)}{\tau_1-\tau}\,d\tau_1,$$

the meaning of the symbol $l_{\delta(\tau)}$ is obvious. Consider the second integral:

$$\int_{l_{\delta(\tau)}} \frac{\varphi(\tau,\ \tau_1)}{\tau_1-\tau}\,d\tau_1=\int_{l_{\delta(\tau)}} \frac{\varphi(\tau,\ \tau_1)-\varphi(\tau,\ \tau)}{\tau_1-\tau}\,d\tau_1+\varphi(\tau,\ \tau)\int_{l_{\delta(\tau)}} \frac{d\tau_1}{\tau_1-\tau}.$$

Since the first integral is improper, it can be made arbitrarily small if δ is sufficiently small; the second integral also tends to zero because of the smoothness of the contour. From the integrability of the function $\omega(\tau, z)$ it follows that $|I_\delta| < \varepsilon$, where ε is an arbitrary small number. A similar estimate holds for the integral $I_{1\delta}$. The modulus of the difference

$$|I - I_1| = |I_\delta - I_{1\delta}| < |I_\delta| + |I_{1\delta}|$$

can be made arbitrarily small, which leads to the equality

$$I = I_1. \tag{2.15}$$

We now turn to the consideration of the case when both integrals are singular:

$$I(t) = \frac{1}{\pi i} \int_L \frac{d\tau}{\tau - t} \frac{1}{\pi i} \int_L \frac{\varphi(\tau, \tau_1)}{\tau_1 - \tau} d\tau_1,$$

$$I_1(t) = \frac{1}{\pi i} \int_L d\tau_1 \frac{1}{\pi i} \int_L \frac{\varphi(\tau, \tau_1)}{(\tau - t)(\tau_1 - \tau)} d\tau.$$

It should be noted that both integrals have sense. To verify this, we introduce the auxiliary function

$$\chi(\tau) = \frac{1}{\pi i} \int_L \frac{\varphi(\tau, \tau_1)}{\tau_1 - \tau} d\tau_1.$$

By the Plemelj-Privalov theorem,* this function belongs to the class H-L, and hence its Cauchy-type integral exists. We next rearrange the integrand in the second integral in the form

$$\frac{1}{(\tau - t)(\tau_1 - \tau)} = \frac{1}{\tau_1 - t} \left[\frac{1}{\tau - t} - \frac{1}{\tau - \tau_1} \right]$$

and introduce the auxiliary function

$$\omega(\zeta, \tau_1) = \frac{1}{\pi i} \int_L \frac{\varphi(\tau, \tau_1)}{\tau - \zeta} d\tau.$$

By means of this function the integral $I_1(t)$ is represented as an improper integral.

Let us extend the integrals $I(t)$ and $I_1(t)$ to the plane of the variable z (by a change from t to z). Denote these integrals by $I(z)$ and $I_1(z)$. It appears from what has been proved before that $I(z) = I_1(z)$. The following equalities are then obvious:

$$I^+(t) = I_1^+(t), \quad I^-(t) = I_1^-(t).$$

* Assuming φ to be a function of two arguments.

By applying the Sokhotskii-Plemelj formulas,* we obtain

$$I(t)=\frac{1}{2}\left[I_1^+(t)+I_1^-(t)\right].$$

We form the auxiliary function

$$\psi(z,\ \tau_1)=\frac{1}{\pi i}\int_L \frac{\varphi(\tau,\ \tau_1)}{\tau-z}\,d\tau-\frac{1}{\pi i}\int_L \frac{\varphi(\tau,\ \tau_1)}{\tau-\tau_1}\,d\tau.$$

With its help we obtain

$$I_1(z)=\frac{1}{\pi i}\int_L \frac{\psi(z,\ \tau_1)}{\tau_1-z}\,d\tau_1.$$

Let us now determine the limiting values $I_1^+(t)$ and $I_1^-(t)$ noting that the limiting value of the density function $\psi(z,\ \tau_1)$ is different on the different sides of the contour; by adding them together, we obtain

$$\frac{1}{2}\left[I_1^+(t)+I_1^-(t)\right]=\frac{1}{2}\left[\psi^+(t,\ t)-\psi^-(t,\ t)\right]+$$
$$+\frac{1}{2\pi i}\int_L \frac{\psi^+(t,\ \tau_1)+\psi^-(t,\ \tau_1)}{\tau_1-t}\,d\tau_1.$$

We transform the right-hand side of this expression by using the Sokhotskii-Plemelj formulas:

$$\frac{1}{2}\left[\psi^+(t,\ t)-\psi^-(t,\ t)\right]=\varphi(t,\ t),$$

$$\frac{1}{2}\left[\psi^+(t,\ \tau_1)+\psi^-(t,\ \tau_1)\right]=\frac{1}{\pi i}\int_L \frac{\varphi(\tau,\ \tau_1)}{\tau-t}\,d\tau-\frac{1}{\pi i}\int_L \frac{\varphi(\tau,\ \tau_1)}{\tau-\tau_1}\,d\tau=$$
$$=\frac{t-\tau_1}{\pi i}\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)(\tau-\tau_1)}\,d\tau.$$

We thus arrive at the required formula, the Poincaré-Bertrand formula,

$$\int_L \frac{d\tau}{\tau-t}\int_L \frac{\varphi(\tau,\ \tau_1)}{\tau_1-\tau}\,d\tau_1=-\pi^2\varphi(t,\ t)+$$
$$+\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)(\tau_1-\tau)}\,d\tau. \qquad (2.16)$$

Let us consider a special case when the density function φ is a function of only one argument τ_1. It can be shown that the integral on the right-hand side of formula (2.16) vanishes. Then

$$\int_L \frac{d\tau}{\tau-t}\int_L \frac{\varphi(\tau_1)}{\tau_1-\tau}\,d\tau=-\pi^2\varphi(t). \qquad (2.16')$$

* Assuming φ to be a function of two arguments.

We now turn to the discussion of the Cauchy-type integral for an unclosed contour. Let L be a smooth unclosed contour with its ends at points a and b. We fix the sense of description, say from the point a to the point b, and consider the integral

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau-z}\,d\tau. \tag{2.17}$$

Here the function $\varphi(\tau)$ belongs to the class $H(A, \lambda)$ at all points of the contour L including the ends. Integral (2.17) will also be termed Cauchy's integral. This integral, in contrast to integral (2.2), is not a piecewise analytic but an analytic function in the whole plane, with the exception of the contour L. By analogy with the Cauchy-type integral for a closed contour, in the case under consideration we also introduce the concept of a singular value and the concepts of the limiting values from the left and right, $\Phi^+(t)$ and $\Phi^-(t)$, in relation to the sense of integration.

Since integral (2.17) remains unaltered if the contour of integration is complemented in some way to form a closed contour and if the function $\varphi(\tau)$ is set equal to zero, it becomes evident that all the foregoing results, which are of a local nature, hold at the interior points of the contour L. Let us study the behaviour of Cauchy's integrals in the neighbourhood of the ends. By transforming expression (2.17), we obtain

$$\Phi(z)=\frac{\varphi(t)}{2\pi i}\int_L \frac{d\tau}{\tau-z}+\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau-z}\,d\tau=$$

$$=\frac{\varphi(t)}{2\pi i}\ln\frac{b-z}{a-z}+\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau-z}\,d\tau, \tag{2.18}$$

The integral in this equality exists as an improper integral when the points z tend to the points of the contour L (including the ends) and when the points of the contour are directly substituted for z. The singularity for the function $\Phi(z)$ is determined only by the first term. By setting $z=a$ and $z=b$ successively, we obtain the following representations:

$$\begin{aligned}\Phi(z)&=-\frac{\varphi(a)}{2\pi i}\ln(z-a)+\Phi_1(z),\\ \Phi(z)&=\frac{\varphi(b)}{2\pi i}\ln(z-b)+\Phi_2(z).\end{aligned} \tag{2.19}$$

Here $\Phi_1(z)$ and $\Phi_2(z)$ are analytic functions bounded in the neighbourhood of the corresponding ends and tending to definite limits as the point z tends to a or b.

The result obtained immediately provides the answer to the question of the behaviour of the Cauchy-type integral at a point where the density function has a discontinuity of the first kind. Denote

this point by c, and the corresponding limiting values of the density function from the left and right by $\varphi(c-0)$ and $\varphi(c+0)$. By using the previous results, we obtain

$$\Phi(z)=\frac{\varphi(c-0)-\varphi(c+0)}{2\pi i}\ln(z-c)+\Phi_0(z). \tag{2.20}$$

Here $\Phi_0(z)$ is an analytic function bounded in the neighbourhood of the point c and tending to a definite limit as z tends to the point c.

We now turn to the consideration of a more complex case when the density function of the Cauchy-type integral has a singularity at one of the ends of the contour (for definiteness, at the point a):

$$\varphi(t)=\frac{\varphi^*(t)}{(t-a)^\gamma}=\frac{\varphi^{**}(t)}{|t-a|^\alpha} \tag{2.21}$$

Here the function $\varphi^*(t)$ satisfies the H-L condition everywhere on the contour including the point a, $\varphi^{**}(t)$ is a bounded function everywhere, $\gamma=\alpha+i\beta$, $0\leqslant\alpha<1$. By the radical $(t-a)^{-\gamma}$ is understood the boundary value from the left of either branch of the function $(z-a)^{-\gamma}$ in the plane cut along the contour L from the point a to the point b and further to infinity along an arbitrary arc not intersecting the contour L. The limiting value from the right $[(z-a)^{-\gamma}]^-$ is then equal to $(t-a)^{-\gamma}e^{-2\pi i\gamma}$.

Let us show that in the neighbourhood of the point a the following estimate holds:

$$|\Phi(z)|<\frac{C}{|z-a|^\gamma}\quad(\alpha<\gamma<1). \tag{2.22}$$

We begin with the consideration of the special case when $\varphi(t)=1$. On the basis of the Sokhotskii-Plemelj formulas, valid, as noted above, at all interior points of the contour L, we state that

$$\Phi^+(t)-\Phi^-(t)=(t-a)^{-\gamma}\left(\Phi(z)=\frac{1}{2\pi i}\int\limits_L\frac{d\tau}{(\tau-a)^\gamma(\tau-z)}\right). \tag{2.23}$$

We introduce the auxiliary function

$$w(z)=\frac{(z-a)^\gamma}{1-e^{-2\pi i\gamma}}. \tag{2.24}$$

This function is single valued in the plane cut as indicated above. Either of its branches satisfies the equality

$$w^+(t)-w^-(t)=(t-a)^{-\gamma}. \tag{2.25}$$

Based on the foregoing, we may write the following relation:

$$[\Phi(z)-w(z)]^+=[\Phi(z)-w(z)]^-, \tag{2.26}$$

which is fulfilled everywhere on L except at the point a. At the same

time, since we have the estimate

$$|\Phi(z)-w(z)|<\frac{C}{|z-a|^{\gamma}}, \tag{2.27}$$

the possible singularity at the point a is removable.

From the above discussion it follows that

$$\begin{aligned}\Phi(z)&=\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}(z-a)^{-\gamma}+\Phi_0(z),\\ \Phi^+(t)&=\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}(t-a)^{-\gamma}+\Phi_0(t),\\ \Phi^-(t)&=\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}(t-a)^{-\gamma}+\Phi_0(t),\end{aligned} \tag{2.28}$$

where $\Phi_0(z)$ is an analytic function bounded in the neighbourhood of the point a. For the singular value, we have the representation

$$\Phi(t)=\frac{1}{2}[\Phi^+(t)+\Phi^-(t)]=\frac{\cot\gamma\pi}{2i}(t-a)^{-\gamma}+\Phi_0(t). \tag{2.29}$$

We now turn to the consideration of the general case. After simple manipulation, we obtain an expression for the function $\Phi(z)$ in the form of a sum of two integrals:

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi^*(\tau)\,d\tau}{(\tau-a)^{\gamma}(\tau-z)}=$$

$$=\frac{1}{2\pi i}\varphi^*(a)\int_L\frac{d\tau}{(\tau-a)^{\gamma}(\tau-z)}+\frac{1}{2\pi i}\int_L\frac{[\varphi^*(\tau)-\varphi^*(a)]}{(\tau-a)^{\gamma}(\tau-z)}d\tau. \tag{2.30}$$

The first integral has been investigated above. It can be shown that the second integral is less in modulus than the function $\frac{C}{|z-a|^{\alpha_0}}$ $(\alpha-\lambda<\alpha_0<\alpha)$, where λ is the index of the class H-L for the function $\varphi^*(t)$.

We state the final results as follows. The Cauchy-type integral with the density function (2.21) in the neighbourhood of the point a is representable in the form

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi^*(\tau)\,d\tau}{(\tau-a)^{\gamma}(\tau-z)}=\frac{\varphi^*(a)\,e^{i\gamma\pi}}{2i\sin\gamma\pi}(z-a)^{-\gamma}+\Phi_0(z). \tag{2.31}$$

If $\alpha=0$, the function $\Phi_0(z)$ is holomorphic in the neighbourhood of the point a in the plane cut as indicated above and tends to a definite limit when z tends to the point a. In the general case when $\alpha>0$, we have the estimate

$$|\Phi_0(z)|<\frac{C}{|z-a|^{\alpha_0}}\quad(\alpha-\lambda<\alpha_0<\alpha). \tag{2.32}$$

3. Riemann Boundary Value Problem

Let L denote, as before, a smooth closed contour and let $G(t)$ be a given continuous function not vanishing on it. The *index* $\varkappa$ *of the function* $G(t)$ *on the contour* L is defined as the increment of the argument of the function $G(t)$ on passing once round the contour counterclockwise divided by 2π:

$$\varkappa = \operatorname{Ind} G(t) = \frac{1}{2\pi}[\arg G(t)]_L = \frac{1}{2\pi i}[\ln G(t)]. \tag{3.1}$$

The index may also be represented in the obvious integral form:

$$\varkappa = \frac{1}{2\pi}\int_L d\arg G(t) = \frac{1}{2\pi i}\int_L d\ln G(t). \tag{3.1'}$$

Since the function $G(t)$ is continuous, the increment of its argument must be a multiple of 2π, and in consequence the index is always an integer. It follows from the above formulas that the index of the product of two functions is equal to the sum of the indices of the factors, and the index of the quotient (provided the denominator is not zero) is equal to the difference of the indices of the divident and divisor.

In the case when the function $G(t)$ is differentiable and is the boundary value from the outside or inside of an analytic function, it may be stated, on the basis of the equalities

$$\varkappa = \frac{1}{2\pi i}\int_L d\ln G(t) = \frac{1}{2\pi i}\int_L \frac{G'(t)}{G(t)}\,dt, \tag{3.1''}$$

that the index is numerically equal to the number of zeros of the function whose boundary value is the function $G(t)$.* In the case when the function $G(t)$ is the boundary value from the inside of an analytic function the sign of the index is positive, otherwise it is negative.

We introduce into consideration functions $G(t)$ and $g(t)$ $[G(t) \neq 0]$ satisfying the H-L condition and given on a closed smooth contour L. The *Riemann problem* consists in finding a piecewise analytic function $\Phi(z)$ (the line of discontinuity is the contour L) satisfying the limiting relation

$$\Phi^+(t) = G(t)\,\Phi^-(t) + g(t). \tag{3.2}$$

The function $G(t)$ is called the *coefficient* and $g(t)$ the *free term of the Riemann problem.* The index of the function $G(t)$ is called the *index of the* corresponding *Riemann problem.* If $g(t) = 0$, the Riemann problem is said to be *homogeneous.*

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* Since the last integral is the logarithmic residue (A. I. Markushevich [1]).

Consider the simplest case when $G(t) = 1$. The solution of the Riemann problem is at once representable by the Cauchy-type integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{\tau - z} d\tau; \tag{3.3}$$

the proof can be obtained directly from the Sokhotskii-Plemelj formulas (2.9).

We now turn to the consideration of the homogeneous problem and assume that it is solvable, i.e., there is a solution $\Phi(z)$ that is not identically zero. Denote by N^+ the number of zeros of the function $\Phi^+(z)$, and by N^- accordingly the number of zeros of $\Phi^-(z)$. Let us calculate the index for the functions entering into the limiting relation

$$\Phi^+(t) = G(t)\,\Phi^-(t). \tag{3.4}$$

We obtain

$$N^+ + N^- = \varkappa. \tag{3.5}$$

Since the left-hand side of (3.5) is a non-negative number, this equality at once enables us to draw the following conclusions regarding the solvability of the homogeneous Riemann problem.

(1) For the homogeneous Riemann problem to be solvable, the index $\varkappa$ must be non-negative.

(2) If $\varkappa > 0$, the functions $\Phi^+(z)$ and $\Phi^-(z)$ have altogether $\varkappa$ zeros.

(3) If $\varkappa = 0$, $\ln G(t)$ is a single-valued function, and the functions $\ln \Phi^+(z)$ and $\ln \Phi^-(z)$ are analytic in D^+ and D^-, respectively. By taking the logarithm of the boundary condition (3.4) [choosing either branch for the function $\ln G(t)$], we arrive at the relation

$$\ln \Phi^+(t) = \ln \Phi^-(t) + \ln G(t). \tag{3.6}$$

We thus obtain a non-homogeneous Riemann problem for the function $\ln \Phi(z)$ with the coefficient $G_1(t) = 1$. Its solution is representable by means of formula (3.3):

$$\ln \Phi(z) = \frac{1}{2\pi i} \int_L \frac{\ln G(\tau)}{\tau - z} d\tau = \Gamma(z).$$

The required solution is of the form

$$\Phi^+(z) = Ce^{\Gamma^+(z)}, \quad \Phi^-(z) = Ce^{\Gamma^-(z)}. \tag{3.7}$$

Thus, if we strictly follow the restrictions introduced above [the analyticity of the function $\Phi(z)$ in the entire region D^- including infinity], we find, on the basis of formulas (3.7) [provided $\Gamma^-(\infty) = 0$], that the solution of the homogeneous problem is always zero since the non-trivial solution furnished by formulas (3.7) is equal to C at infinity.

The foregoing enables us to obtain the following result. Let $\varphi(t)$ be a given function on a closed contour belonging to the class H-L and not vanishing on it (if $\varkappa = 0$). It may be represented as the quotient of functions that are the boundary values of analytic functions, respectively, in D^+ and D^- (with the exception of the point at infinity), not vanishing in these regions. The indicated functions are determined by formulas (3.7).

(4) If $\varkappa > 0$, we rewrite the boundary condition (3.4) as

$$\Phi^+(t) = t^{\varkappa}[t^{-\varkappa}G(t)]\,\Phi^-(t) = t^{\varkappa}G_1(t)\,\Phi^-(t).$$

For definiteness, we assume that zero belongs to the region D^+. Since the index of the function $G_1(t)$ is zero, it may be represented, on the basis of the foregoing, in the form of a quotient:

$$G_1(t) = \frac{e^{\Gamma^+(t)}}{e^{\Gamma^-(t)}}, \quad \Gamma(z) = \frac{1}{2\pi i}\int\limits_L \frac{\ln[\tau^{-\varkappa}G(\tau)]}{\tau - z}\,d\tau.$$

The boundary condition is therefore rewritten as

$$\frac{\Phi^+(t)}{e^{\Gamma^+(t)}} = t^{\varkappa}\frac{\Phi^-(t)}{e^{\Gamma^-(t)}}. \tag{3.8}$$

The left-hand side of equality (3.8) is the boundary value of a function analytic in D^+, and the right-hand side of a function analytic in D^-, with the exception of the point at infinity where it has a pole of order not higher than $\varkappa$. Based on the generalized Liouville theorem, we conclude that the general solution of the boundary value problem (3.8) is representable by the following formulas:

$$\Phi^+(z) = e^{\Gamma^+(z)}P_{\varkappa-1}(z), \quad \Phi^-(z) = e^{\Gamma^-(z)}z^{-\varkappa}P_{\varkappa-1}(z). \tag{3.9}$$

Here $P_{\varkappa-1}(z)$ is an arbitrary polynomial of degree not higher than $\varkappa - 1$.

The solution furnished by formulas (3.9) is called the general solution of the homogeneous Riemann problem. If it is assumed that $\Phi^-(\infty) \neq 0$, the polynomial $P(z)$ in (3.9) must be of degree $\varkappa$.*

Thus, if $\varkappa > 0$, the homogeneous Riemann problem admits $\varkappa$ (or correspondingly $\varkappa + 1$) linearly independent solutions:**

$$\Phi_k^+(z) = z^k e^{\Gamma^+(z)}, \quad \Phi_k^-(z) = z^{k-\varkappa}e^{\Gamma^-(z)}. \tag{3.9'}$$

We introduce the concept of a canonical function of the Riemann problem. A piecewise analytic function $X(z)$ representable in the form

$$X^+(z) = e^{\Gamma^+(z)}, \quad X^-(z) = z^{-\varkappa}e^{\Gamma^-(z)},$$

* This assumption will be further considered fulfilled.

** In solving boundary value problems having a definite physical meaning, there arise as a rule certain considerations permitting a unique determination of the polynomial $P_{\varkappa}(z)$.

will be termed a *canonical function of the Riemann problem.* When $\varkappa \geqslant 0$, the canonical function is a particular solution of the homogeneous problem and the general solution may be written as

$$\Phi(z) = X(z)\, P_{\varkappa}(z).$$

When $\varkappa < 0$, the canonical function also satisfies the boundary relation (3.4), but it has a pole of order $-\varkappa$ at infinity.

The introduction of a canonical function makes it possible to extend the foregoing result (p. 49) to functions of arbitrary index. It can be shown that every function belonging to the class H-L and not vanishing on a closed contour may be represented as

$$\varphi(t) = \frac{X^+(t)}{X^-(t)}. \tag{3.10}$$

Here $X(z)$ is the canonical function of the Riemann problem $\Phi^+(t) = \varphi(t)\, \Phi^-(t)$.

We proceed to the solution of the non-homogeneous problem. It will be recalled that the Riemann problem in this case consists in constructing a piecewise analytic function $\Phi(z)$ satisfying relation (3.2).

Let $\varkappa$ be the index of the function $G(t)$ and let $X(z)$ be the canonical function of the stated problem [when $g(t) = 0$]. We then have the equality

$$G(t) = \frac{X^+(t)}{X^-(t)}. \tag{3.11}$$

The limiting relation (3.2) may be represented as

$$\frac{\Phi^+(t)}{X^+(t)} = \frac{\Phi^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}. \tag{3.12}$$

Consider the auxiliary Riemann problem

$$\Psi^+(t) = \Psi^-(t) + \frac{g(t)}{X^+(t)}, \tag{3.13}$$

whose solution is given by the formula

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}. \tag{3.14}$$

We rearrange the boundary condition (3.12) taking into account (3.14):

$$\frac{\Phi^+(t)}{X^+(t)} - \Psi^+(t) = \frac{\Phi^-(t)}{X^-(t)} - \Psi^-(t). \tag{3.15}$$

The left-hand side of equality (3.15) is the boundary value of a function analytic in D^+, and the right-hand side of a function analytic in D^-, with the possible exception of the point at infinity. In the case when $\varkappa < 0$, the quotient $\Phi^-(z)/X^-(z)$ vanishes at infinity. Since the function $\Psi^-(z)$ also vanishes at infinity, we con-

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clude, on the basis of Liouville's theorem, that the expressions on both sides of equality (3.15) are identically zero, from which it follows that

$$\Phi(z) = X(z)\,\Psi(z). \tag{3.16}$$

If, however, $\varkappa \geqslant 0$, the quotient $\Phi^-(z)/X^-(z)$ is the boundary value of a function analytic everywhere in D^- except at the point at infinity, where it has a pole of order $\varkappa$. We therefore conclude from the generalized Liouville theorem that the expressions on both sides of equality (3.15) are identically equal to a certain polynomial of degree $\varkappa$. The solution in this case is of the form

$$\Phi(z) = X(z)\,[\Psi(z) + P_\varkappa(z)]. \tag{3.17}$$

Solutions (3.16) and (3.17) may be represented analytically by the single expression (3.17) implying that the polynomial is absent when $\varkappa < 0$.

It is necessary to make an additional analysis of the solution when $\varkappa < 0$. The function $X(z)$ has a pole of order $-\varkappa$ at infinity, and the function $\Psi(z)$ has, in general, a zero of order unity. Hence, the product $X(z)\,\Psi(z)$ [equal to the function $\Phi(z)$] has a pole of order $-\varkappa - 1$ at infinity. Consequently, the non-homogeneous Riemann problem is unsolvable when $\varkappa + 1 < 0$. It is solvable only when the free term satisfies certain conditions ensuring the analyticity of the function $\Phi(z)$ at the point at infinity. We expand the function $\Psi(z)$ in a series:

$$\Psi^-(z) = \sum_{k=1}^{\infty} c_k z^{-k}, \qquad c_k = -\frac{1}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)}\,\tau^{k-1}\,d\tau.$$

It is obvious that the function $\Phi(z)$ is analytic at infinity if $c_k = 0$ $(k = 1, 2, \ldots, -\varkappa - 1)$.

By summing up the foregoing, we obtain the following theorem. The non-homogeneous Riemann problem (if it is solvable) has a solution representable by formula (3.17). When $\varkappa \geqslant 0$, the Riemann problem is always solvable, and when $\varkappa < 0$ it is solvable only if the following conditions are fulfilled:

$$\int_L \frac{g(\tau)}{X^+(\tau)}\,\tau^{k-1}\,d\tau = 0 \quad (k = 1, 2, \ldots, -\varkappa - 1). \tag{3.18}$$

4. Singular Integral Equations

A *singular integral equation* is an integral equation of the form

$$K\varphi = a(t)\,\varphi(t) + \frac{1}{\pi i}\int_L \frac{M(t,\tau)}{\tau - t}\,\varphi\tau\,d\tau = f(t). \tag{4.1}$$

Here L is a smooth closed contour containing, for definiteness, zero, and the functions $a(t)$, $M(t, \tau)$, and $f(t)$ belong to the class H-L;

the function $M(t, \tau)$ belongs to this class with respect to both arguments. We introduce the following notation:

$$b(t) = M(t, t), \quad \frac{1}{\pi i} \frac{M(t, \tau) - M(t, t)}{\tau - t} = k(t, \tau).$$

It is obvious that the function $b(t)$ also belongs to the class **H-L** and we have the estimate

$$|k(t, \tau)| < \frac{A}{|\tau - t|^{1-\lambda}} \quad (0 < \lambda \leqslant 1).$$

Equation (4.1) is rewritten as

$$K\varphi = a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau + \int_L k(t, \tau)\varphi(\tau) d\tau = f(t). \quad (4.1')$$

Equation (4.1′) is called the complete singular equation represented in standard form. The equation is said to be *homogeneous* if $f(t) = 0$. The operator

$$K^0\varphi = a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau$$

is called the *characteristic part* of the singular equation, and the operator

$$k\varphi = \int_L k(t, \tau)\varphi(\tau) d\tau$$

is the *regular part* of the singular equation.

In the notation adopted Eq. (4.1) is written as

$$K\varphi = K^0\varphi + k\varphi = f. \quad (4.1'')$$

A *companion* (*transposed*) or *adjoint* equation to Eq. (4.1) is an equation of the form

$$K'\psi = a(t)\psi(t) - \frac{1}{\pi i} \int_L \frac{b(\tau)\psi(\tau)}{\tau - t} d\tau + \int_L k(\tau, t)\psi(\tau) d\tau = 0. \quad (4.2)$$

It should be noted that the companion equation to the characteristic one is not, in general, a characteristic equation since it involves an additional regular term.

Let K be a singular integral operator of the form of (4.1′) and let K' be the corresponding companion operator (4.2). By direct substitution (using, as proved in Sec. 2, the possibility of interchanging the order of integration in the case when one of the integrals is regular) we ascertain that the following equality is fulfilled identically:

$$\int_L \psi K\varphi \, d\tau = \int_L \varphi K'\psi \, d\tau. \quad (4.3)$$

Let K_1 and K_2 be singular operators of the form of (4.1′) with coefficients in the characteristic part, respectively, equal to $a_1(t)$, $b_1(t)$ and $a_2(t)$, $b_2(t)$. By using the Poincaré-Bertrand interchange formulas (2.16′), it can be shown that the composition of singular operators $K = K_1K_2$ is also a singular operator with the following coefficients in the characteristic part:

$$\begin{aligned} a(t) &= a_1(t)\,a_2(t) + b_1(t)\,b_2(t), \\ b(t) &= a_1(t)\,b_2(t) + a_2(t)\,b_1(t). \end{aligned} \tag{4.4}$$

It follows from formulas (4.4) that the characteristic part of a composition of singular operators is independent of the regular parts of each of them and is determined only by their characteristic parts. It should be noted that the characteristic part of a composition of singular operators is independent of its order.

Direct substitution proves the equality

$$(K_2K_1)' = K_1' \cdot K_2'. \tag{4.5}$$

Consider the simplest singular equation, viz. the characteristic equation

$$K^0\varphi = a(t)\,\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t). \tag{4.6}$$

We introduce a piecewise analytic function:

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau.$$

Substituting the expressions for $\varphi(t)$ and $\Phi(t)$, according to (2.9′), in Eq. (4.6), we arrive at the auxiliary Riemann problem:

$$\begin{gathered} \Phi^+(t) = G(t)\,\Phi^-(t) + g(t), \\ G(t) = \frac{a(t) - b(t)}{a(t) + b(t)}, \qquad g(t) = \frac{f(t)}{a(t) + b(t)}. \end{gathered} \tag{4.7}$$

The index of this problem will be termed the index of the integral equation (4.6).

In the following discussion we shall only consider so-called normal equations assuming that the inequality $a(t) \pm b(t) \neq 0$ holds everywhere on the contour L.

On the basis of (2.9′) and (3.17) we obtain the solution of the auxiliary Riemann problem, from which we have the required function:

$$\varphi(t) = a(t) f(t) - \frac{b(t) Z(t)}{\pi i} \int_L \frac{f(\tau)}{Z(\tau)} \frac{d(\tau)}{\tau - t} + b(t) Z(t) P_{\varkappa-1}(t),$$

$$Z(t) = [a(t) + b(t)] X^+(t) = [a(t) - b(t)] X^-(t) = \frac{e^{\Gamma(t)}}{t^{\varkappa/2}}, \tag{4.8}$$

$$\Gamma(t) = \frac{1}{2\pi i} \int_L \frac{\ln\left[\tau^{-\varkappa} \frac{a(\tau) - b(\tau)}{a(\tau) + b(\tau)}\right]}{\tau - t} d\tau.$$

Thus, the answer to the question of the existence of the solution for the characteristic singular integral equation and its construction follow from the corresponding Riemann problem.

We now turn to the solution of the companion equation to the characteristic one:

$$K^{0\prime}\psi = a(t)\psi(t) - \frac{1}{\pi i} \int_L \frac{b(\tau)\psi(\tau)}{\tau - t} d\tau = 0; \tag{4.9}$$

By using the substitution $\omega(t) = b(t)\psi(t)$, we transform it to the characteristic equation in the auxiliary function $\omega(t)$:

$$a(t)\omega(t) - \frac{b(t)}{\pi i} \int_L \frac{\omega(\tau)}{\tau - t} d\tau = 0;$$

by means of the piecewise analytic function

$$\Omega(z) = \frac{1}{2\pi i} \int_L \frac{\omega(\tau)}{\tau - z} d\tau$$

we transform to the auxiliary Riemann problem. Its index $\varkappa'$ is equal to the negative of the index $\varkappa$ of the original problem since the coefficient of the Riemann problem for the function $\Omega(z)$ is $\frac{a(t)+b(t)}{a(t)-b(t)}$. Comparison between the solutions of the Riemann problem for the original (characteristic) and the adjoint equation enables us to formulate some statements concerning their solvability in the form accepted in the theory of Fredholm integral equations.

The homogeneous singular characteristic equation and the companion equation are never solvable simultaneously; either they are both unsolvable ($\varkappa = 0$) or the one for which the index is positive is solvable. The difference between the numbers of their linearly independent solutions is equal to $|\varkappa|$.

The non-homogeneous characteristic equation (4.6) is always solvable whatever the right-hand side if $\varkappa \geqslant 0$. If $\varkappa < 0$, the auxiliary Riemann problem is solvable only when the special conditions

(3.18) are fulfilled. It can be shown that these conditions transform to

$$\int_L \psi_i(t) f(t)\, dt = 0 \qquad (i = 1,\ 2,\ \ldots,\ -\varkappa).$$

Here $\psi_i(t)$ is the complete system of eigenfunctions of the homogeneous companion equation.

We now direct our attention to the question of the regularization of singular operators. Let K_1 and K_2 be singular operators of the form of (4.1′). If the operator K_2 is such that the composition K_2K_1 is a regular operator (i.e. ,it involves no singular integral), the operator K_2 is called the *regularizator of the operator* K_1. It is obvious that if the operator K_2 is the regularizator of the operator K_1, then the operator K_1 is the regularizator of the operator K_2. It follows from formulas (4.4) that the characteristic part of the regularizing operator must be of the form

$$K_2^0\varphi = a(t)\,\varphi(t) - \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

where $a(t)$ and $b(t)$ are the coefficients in the characteristic part of the operator K_1 being regularized. The regularizing operator will be denoted by $\widetilde{K}$.

Consider an equation of the form of (4.1″), $K\varphi = f$. By applying the operator $\widetilde{K}$ to both sides of the equation, we obtain the equation

$$\widetilde{K}K\varphi = \widetilde{K}f, \tag{4.10}$$

which is regular. Thus, the function $\varphi(t)$ satisfies both the singular and the regular equation. The foregoing procedure is called *left-hand regularization.*

If, instead of the unknown function $\omega(t)$,we introduce an auxiliary function $\varphi(t)$ defined by the relation $\varphi(t) = \widetilde{K}\omega$ into Eq. (4.1″), we also arrive at a regular equation, namely

$$K\widetilde{K}\omega = f. \tag{4.11}$$

In this case, instead of the singular equation for the unknown function we have obtained a regular equation for the auxiliary function $\omega(t)$. By solving this equation, we obtain the unknown function by means of the operator $\widetilde{K}$ ($\varphi = \widetilde{K}\omega$). This procedure is called *right-hand regularization.*

Since the theory of regular (Fredholm) equations is thoroughly developed (see Sec. 1), the relation between the solutions of Eqs. (4.10), (4.11) and the original complete singular equation (4.1) enables one to construct the theory of singular integral equations (Noether's theorems).

In the process of passing from the singular to the regular equation (whether by left-hand or right-hand regularization) it is possible that certain solutions may be lost and that functions may appear which are a solution of the regular equation, but are not a solution of the singular equation. In other words, the resulting regular equation may not be equivalent to the original one.

We first consider this question in reference to left-hand regularization. Equation (4.10) may be represented in the form

$$\widetilde{K}\,(K\varphi-f)=0. \qquad (4.12)$$

Since the operator $\widetilde{K}$ is homogeneous, every solution of Eq. (4.1) satisfies Eq. (4.12). Left-hand regularization therefore involves no loss of solutions. In the case when the operator $\widetilde{K}^*$ has no eigenfunctions ($\varkappa > 0$), left-hand regularization is obviously equivalent. The presence of eigenfunctions of the operator $\widetilde{K}$ makes possible the appearance of additional solutions, which are, in general, solutions of the equation

$$K\varphi=f+\sum\alpha_j\omega_j,$$

where ω_j are the eigenfunctions and α_j are arbitrary constants.

In right-hand regularization (in contrast to left-hand regularization) there may be a loss of solutions since the equation $\widetilde{K}\omega = \varphi_0$ [φ_0 (t) is a solution of Eq. (4.1)] is not always solvable. Regularization is therefore equivalent when this equation is solvable with any right-hand side, which happens when $\varkappa \leqslant 0$.

The foregoing results enable us to prove the basic alternatives of the theory of singular integral equations.

(1) The number of linearly independent solutions of a singular equation is finite. The proof follows from the fact that no loss of solutions occurs in left-hand regularization, and the number of solutions of the resulting Fredholm equation is finite.

(2) A necessary and sufficient condition for the solvability of Eq. (4.1) is the fulfilment of the following equalities:

$$\int\limits_L f(t)\,\psi_j(t)\,dt=0 \quad (j=1,\ 2,\ \ldots,\ n). \qquad (4.13)$$

Here ψ_j (t) is the set of linearly independent solutions of the equation $K'\psi = 0$.

The necessity of conditions (4.13) follows immediately from identity (4.3). Indeed,

$$\int\limits_L f(t)\,\psi_j(t)\,dt=\int\limits_L K\varphi(t)\,\psi_j(t)\,dt=\int\limits_L \varphi K'\psi_j\,dt=0 \quad (j=1,2,\ldots,n).$$

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* In the following discussion only the simplest (characteristic) operators, $\widetilde{K} = K^{0'}$, are used as regularizing operators.

The proof of sufficiency is carried out in different ways depending on the sign of the index. We first consider the case when $\varkappa \geqslant 0$. The regularizing operator $\widetilde{K}$ has the index $-\varkappa \leqslant 0$, and hence has no eigenfunctions. Equation (4.11) is therefore equivalent to the original one, and Eqs. (4.11) and (4.1) are both solvable or unsolvable. According to the Fredholm alternative, Eq. (4.11) is solvable if the following conditions are fulfilled:

$$\int_L \chi_j \widetilde{K} f\, dt = 0. \tag{4.14}$$

Here $\chi_j(t)$ are the solutions of the equation $K'\widetilde{K}'\chi = 0$, companion to Eq. (4.11). We transform Eq. (4.14):

$$\int_L \chi_j \widetilde{K} f\, dt = \int_L f \widetilde{K}' \chi_j\, dt = 0.$$

Let us consider the equation $K'\widetilde{K}'\chi = 0$ as an equation with the operator K' and the unknown function $\widetilde{K}'$. The function $\widetilde{K}'\chi$ is then the eigenfunction of the operator K'; denoting it by $\psi_j(t)$, we arrive at condition (4.13).

When $\varkappa < 0$, we apply right-hand regularization. By substituting $\varphi = \widetilde{K}\omega$, we arrive at a Fredholm equation

$$K\widetilde{K}\omega = f, \tag{4.15}$$

equivalent to the original equation (4.1). The conditions for the solvability of Eq. (4.15) are of the form

$$\int_L f(t)\, \chi_j(t)\, dt = 0,$$

where $\chi_j(t)$ is any one of the solutions of the equation $\widetilde{K}'K'\chi = 0$, companion to Eq. (4.15). By considering it as an equation with the operator $\widetilde{K}'$ and the unknown function $K'\chi$, and noting that $\widetilde{K}'$ is an operator with negative index (no eigenfunctions), we arrive at the equation $K'\chi = 0$. Consequently, $\chi_j(t)$ is the eigenfunction of the operator K'. Denoting it, as before, by $\psi_j(t)$, we arrive at the required orthogonality relation.

The difference between the number of linearly independent solutions (n) of the singular equation $K\varphi = 0$ and the number of linearly independent solutions (n') of the companion equation $K'\psi = 0$ is equal to the index of the equation:

$$n - n' = \varkappa. \tag{4.16}$$

Assume $\varkappa \geqslant 0$. The regularizing operator is taken to be $K^{0\prime}$. The Fredholm equation $K^{0\prime}K\varphi = 0$ is then equivalent to the original one and so also has n solutions. Accordingly, the companion equation

$K'K^0\psi = 0$ also has exactly n solutions. The resulting equation is equivalent to the equation

$$K^0\psi = \alpha_1\psi_1 + \ldots + \alpha_{n'}\psi_{n'},$$

where $\psi_j(t)$ are the eigenfunctions of the operator K' and α_j are arbitrary constants. Since $\varkappa \geqslant 0$, the last equation is solvable with any right-hand side, and its solution is of the form

$$\psi(t) = \sum_{j=1}^{n'} \alpha_j R\psi_j + \sum_{j=1}^{\varkappa} c_j\varphi_j(t),$$

where R is the symbolic form of writing the solution of the singular equation with the appropriate right-hand side. Let us prove that all functions appearing on the right-hand side are linearly independent. Assume that the relation

$$\sum_{j=1}^{n'} \alpha_j R\psi_j + \sum_{j=1}^{\varkappa} c_j\varphi_j(t) \equiv 0,$$

is fulfilled if only for one $\alpha_j \neq 0$. Operating on this equality with K^0, we find that $\sum_{j=1}^{n'} \alpha_j\psi_j \equiv 0$, which is impossible because of the linear independence of the functions $\psi_j(t)$. The case when all $\alpha_j = 0$ leads to a linear dependence of the functions $\varphi_j(t)$, which is also excluded.

Thus, the integral equation $K'K^0\psi = 0$ has $n' + \varkappa$ solutions, and hence $n = n' + \varkappa$.

The consideration of the case $\varkappa < 0$ is not required since the property of operators to be companion is reciprocal, and the companion operator must be taken as the original one since its index is $\varkappa' = -\varkappa > 0$.

It should be noted that in constructing the theory of singular integral equations much use has been made of specific properties of special singular operators. Below we shall consider the question in general terms within the framework of functional analysis, and a number of new results will be obtained in the course of the discussion. The advantage of this approach will come to light in studying two-dimensional singular operators (Secs. 7, 8).

The study will be carried out in Banach space denoted by H (see L. A. Lyusternik, V. I. Sobolev [1]). We give some definitions. An operator is said to be *bounded* if it transforms any bounded sequence into a bounded one. An operator is called *completely continuous* if it transforms any bounded sequence into a compact one (i.e., a sequence from which a convergent subsequence can be chosen).

It can be shown (N. I. Muskhelishvili [4]) that singular integrals are operators bounded in space with the norm

$$\| \varphi \| = \max |\varphi(t)| + \max \frac{|\varphi(t_2) - \varphi(t_1)|}{|t_2 - t_1|^{\mu}},$$

where μ is the index of the class H-L for the density function. Regular integrals are completely continuous operators.

Within the framework of functional analysis it is possible to give a general formulation of the problem of regularizing an operator equation (when the operator itself is bounded)

$$A\varphi = f. \tag{4.17}$$

A bounded operator B is called a regularizator for the operator A if, by applying it to both sides of Eq. (4.17), we arrive at the equation

$$BA\varphi = (I + T)\varphi = Bf, \tag{4.18}$$

where I is the identity operator, T is a completely continuous operator. The equation thus obtained is called a *Fredholm equation.* The non-trivial solutions of homogeneous operator equations will be further called the zeros of the corresponding operators. The *index of an operator equation* is the difference in the number of zeros of the basic and adjoint equations.

Below are given some of the results following directly from the above discussion. The number of zeros of Eq. (4.17) admitting regularization is bounded since no loss of solutions may occur and the number of zeros of the Fredholm operator is finite (in accordance with the terminology adopted previously the proposed regularization is left-hand regularization). The use of the concept of an adjoint operator enables us to obtain necessary conditions for the solvability of Eq. (4.17). We have

$$(f, \psi_j) = (A\varphi, \psi_j) = (\varphi, A^* \varphi_j) = 0, \quad (j = 1, 2, \ldots, n^*), \tag{4.19}$$

where ψ_j is the complete set of zeros of the adjoint equation.

It is more difficult to prove that conditions (4.19) are also sufficient conditions for solvability, this being the case when the regularizing operator exists. Denote by H_0 the set of zeros of the operator A. We shall show that this set is a subspace. Let φ_k $(k = 1, 2, \ldots, n)$ be the elements of the set H_0. It is obvious that $\sum_{k=1}^{n} c_k \varphi_k \in H_0$. Let the elements φ_k $(\varphi_k \in H_0)$ tend to φ_0. By virtue of the boundedness of the operator A, we have $A\varphi_0 = \lim A\varphi_k = 0$, i.e., $\varphi_0 \in H_0$. In exactly the same way it can be shown that the set of zeros of the operator A^* (which will be denoted by H_0^*) is a subspace.

Let subspaces H_1 and H_1^* be now defined as the orthogonal complements of the subspaces H_0 and H_0^*, respectively. We now consider

Eq. (4.17) only on the elements of the subspace H_1 assuming the right-hand side to belong to the subspace H_1^*. Let us prove that in this case the equation has a solution (i.e., that there exists an inverse operator). First, we shall show that Eq. (4.17) has no more than one solution in H_1. Suppose that it has solutions φ_1 and φ_2. The element $\varphi_1 - \varphi_2$ then belongs to H_1, but, on the other hand, $A(\varphi_1 - \varphi_2) = 0$, and hence $(\varphi_1 - \varphi_2) \in H_0$. Since H_0 and H_1 are orthogonal, it follows that $\varphi_1 - \varphi_2 = 0$. Consequently, there exists an inverse operator on some set H'. We first prove that this set is compact in H_1^* (that is, its closure coincides with H_1^*). Otherwise, there is an element $\omega \in H_1^*$ such that $(f, \omega) = 0$ if $f \in H'$. Let $\varphi = A^{-1}f$. We then arrive at the equality $(A\varphi, \omega) = 0$. This equality must hold for any $\varphi \in H_1$ (since f may be an arbitrary element from the set H'). On the other hand, this equality is obvious when $\varphi \in H_0$. Consequently, we have $(A\varphi, \omega) = 0$ for any $\varphi \in H$. Further $(A\varphi, \omega) = (\varphi, A^*\omega)$, and since φ may be arbitrary we obtain $A^*\omega = 0$, which means $\omega \in H_0^*$. Since ω belongs to H_2^* at the same time, we obtain $\omega = 0$.

We further prove that the operator A^{-1} is bounded in H'. Otherwise, there must exist elements $\varphi_n \in H_1$ such that $\| \varphi_n \| = 1$ and $A\varphi_n \to 0$. By applying the regularizing operator, we obtain

$$BA\varphi_n = \varphi_n + T\varphi_n \to 0.$$

We choose a subsequence φ_{n_k} so that $T\varphi_{n_k}$ tends to some limit φ_0. Then $\varphi_{n_k} \to \varphi_0$. From the boundedness of the operator A it follows that $A\varphi_0 = \lim A\varphi_{n_k} = 0$. Hence, $\varphi_0 \in H_0$. On the other hand, since H_1 is a closed subspace, $\varphi_0 \in H_1$, which leads to the equality $\varphi_0 = 0$, and this is impossible as we have previously assumed $\| \varphi_0 \| = 1$.

It is now possible to prove that the set H' coincides with H_1^*. Let f be an arbitrary element from H_1^*. Since H' is compact, we can construct a sequence of elements f_n converging to f. Let $\varphi_n = A^{-1}f_n$ $(A\varphi_n = f)$. From the boundedness of the operator A^{-1} it follows that there exists a limit of the sequence φ_n $(\tilde{\varphi})$. Further, from the boundedness of the operator A we obtain $A\tilde{\varphi} = \lim A\varphi_n = \lim f_n = f$. Then $f \in H'$, which completes the proof. The foregoing shows that Eq. (4.17) is solvable when conditions (4.19) are fulfilled if there exists a bounded regularizator.

In addition to the ordinary regularization, there is a so-called equivalent regularization for which Eqs. (4.17) and (4.18) have the same solution. Let us show, for example, that the equations $A\varphi = f$ and $A^*A\varphi = A^*f$ are equivalent if the original equation is solvable. Assume the contrary. Besides the function φ_0 $(A\varphi_0 = f)$, there exists then a function φ_1 $(A^*A\varphi_1 = A^*f$ and $A\varphi_1 = f)$. Consider the differ-

ence $A^*A(\varphi_1 - \varphi_0) = 0$ and multiply it by $\varphi_1 - \varphi_0$. We obtain

$$0 = (A^*A(\varphi_1 - \varphi_0, \varphi_1 - \varphi_0)) = (A(\varphi_1 - \varphi_0), A(\varphi_1 - \varphi_0)).$$

Consequently, $A(\varphi_1 - \varphi_0) = 0$, $A\varphi_1 = A\varphi_0$, which leads to contradiction.

We next consider the question of the influence of the completely continuous operator T, additionally introduced into the equation, on the value of the index. It will be proved that the index remains unchanged, i.e., $\operatorname{Ind}(A + T) = \operatorname{Ind} A$.

Note that the operators A and $A + T$ have the same regularizing operator, B. The equations $BA\varphi = 0$ and $A^*B^*\psi = 0$ have an equal number of zeros, which will be denoted by r. Denote further by n, n^*, m, and m^* the number of zeros of the operators A, A^*, B, and B^* respectively. Let φ_j $(j = 1, 2, \ldots, n)$ and χ_j $(j = 1, 2, \ldots, m)$ be the zeros of the operators A and B. It is obvious that the equation $BA\varphi = 0$ is equivalent to the equation

$$A\varphi = \sum_{k=1}^{m} c_k \chi_k, \tag{4.20}$$

where c_k are arbitrary constants. For the solvability of Eq. (4.20), it is necessary, as has been shown above, that the following conditions should be fulfilled:

$$\sum_{k=1}^{m} c_k (\chi_k, \psi_j) = 0 \quad (j = 1, 2, \ldots, n^*). \tag{4.21}$$

Assume that the rank of the matrix $\| (\chi_k, \psi_j) \|$ is s. The solution of Eq. (4.20) then contains $m - s$ constants plus the number of zeros (n) of the operator A. We obtain $r = n + m - s$. Let us now calculate this number proceeding from the equation $A^*B^*\psi = 0$. Consider the equation

$$B^*\psi = \sum_{k=1}^{n^*} \gamma_k \psi_k, \tag{4.22}$$

where γ_k are constants. Since this operator has a regularizator (A^*), the solvability requires the fulfilment of the conditions

$$\sum_{k=1}^{n^*} \gamma_k (\psi_k, \chi_j) = 0 \quad (j = 1, 2, \ldots, m).$$

The matrix $\| (\psi_k, \chi_j) \|$ is adjoint to the matrix $\|(\chi_k, \psi_j)\|$, and hence it is of the same rank s. Thus, we also arrive at the equality

$r = n^* + m^* - s$. We finally obtain

$$n - n^* = m^* - m \quad (\operatorname{Ind} A = -\operatorname{Ind} B). \tag{4.23}$$

Since the introduction of an additional completely continuous operator leaves the right-hand side of equality (4.23) unaltered, we arrive at the required result.*

Let us strengthen the result obtained. It will be shown that a more general equality holds, namely $\operatorname{Ind}(A + C) = \operatorname{Ind} A$, where C is a bounded operator whose norm is less than $\| B \|^{-1}$ (B is, as before, the regularizator of the operator A). We have $BA = I + T$. Then

$$B(A+C) = I + BC + T =$$
$$= (I + BC)\,[I + (I + BC)^{-1}\,T] = (I + BC)\,(I + T_1), \tag{4.24}$$

where T and T_1 are appropriate completely continuous operators. It follows from (4.24) that the operator $A + C$ has the regularizator $(I + BC)^{-1}B$. Because of the restriction ($\| B \| \, \| C \| < 1$) the number of zeros of the latter coincides with that of the operator B. It is also obvious that the operators B^* and $B^*\,(I + C^*B^*)^{-1}$ have the same number of zeros. Thus, we arrive at the equalities

$$-\operatorname{Ind} A = \operatorname{Ind} B = \operatorname{Ind} B\,(I + BC)^{-1} = -\operatorname{Ind}(A + C).$$

Note that if the condition $\| C \| < 1$ is fulfilled, the equation $(I + C)\,\varphi = f$ can be solved by the method of successive approximations.

The foregoing shows that the question of the solvability of an operator equation (with a bounded operator) reduces to establishing the possibility of its regularization and determining all zeros of the companion equation. To construct the complete solution, however, it is necessary to find all zeros of the operator, and this requires, in the first place, the determination of their number. It is therefore important to find the value of the index of the equation (since the number of zeros of the companion equation must be known when establishing the conditions for solvability). It is for this purpose that the possibility of equivalent regularization is examined.

It may happen [it is precisely this case that is encountered for some problems of the theory of elasticity (see Sec. 29)] that the investigation of the equations is completed just by establishing the eigenfunctions of the companion operator, which are necessary for the solvability conditions, and the finding of the eigenfunctions of the original equation is unnecessary since they have no influence on the solution of the original boundary value problem.

* In the course of the discussion we have omitted the proof that the operators A^* and $A^* + T^*$ have a finite number of zeros.

5. Riemann Boundary Value Problem in the Case of Discontinuous Coefficients and Unclosed Contours

In the preceding sections we have considered boundary value problems where the coefficients $G(t)$ and $g(t)$ are continuous functions and the corresponding contours L are closed. The resulting solution (picewise analytic function) is automatically continuously extendible to the contour. By extending the formulation of boundary value problems to unclosed contours, and permitting discontinuities of the first kind for the coefficients $G(t)$ and $g(t)$, we have to introduce, as permissible functions, those having integrable singularities at the points of discontinuity in the coefficients and at the ends of the contours. The necessity of this kind of limitation is associated with considerations of both mathematical and physical (in applications) order. The limitation introduced ensures the uniqueness of solution and the boundedness of energy in boundary value problems having a definite physical meaning.

A problem for an unclosed contour can be reduced to that for a closed contour by a very simple procedure introducing appropriate discontinuities at the ends of arcs. To do this, the ends of arcs must be joined together by some non-intersecting lines so that the resulting system is a single closed contour. It must be assumed that $G(t) = 1$ and $g(t) = 0$ on the auxiliary lines. We thus arrive at a problem for a solid contour with discontinuities in the coefficients at given points. In the following discussion the case of unclosed contours, as a particular case of the general solution, will be the subject of special analysis on account of its exceptional importance in applications.

We now turn to the solution of the Riemann problem for one closed contour when the functions $G(t)$ and $g(t)$ have points of discontinuity of the first kind, and the function $g(t)$ has, in addition, singularities of the form

$$g(t) = \frac{g^*(t)}{|t-t_k|^{\alpha}}, \tag{5.1}$$

where $\alpha < 1$ and $g^*(t)$ belongs to the class H-L.

The solution of the Riemann problem may be sought among functions belonging to different classes. Thus, we may seek solutions bounded at all points of discontinuity in the coefficients or we may seek solutions that are unbounded but integrable at all points of discontinuity in the coefficients. Finally, we may seek solutions bounded in the neighbourhood of some ends and unbounded in the neighbourhood of others. By an unbounded but integrable solution is meant one with singularities of the form

$$|\Phi^{\pm}(t)| < \frac{C}{|t-t_k|^{\alpha}} \quad (\alpha < 1).$$

At points where the coefficient $g(t)$ has a singularity of the form of (5.1) the functions $\Phi^{\pm}(t)$ have a singularity of the same kind as (2.28).

Let us pass to the Riemann problem with a continuous coefficient $G(t)$. We first consider the case when the coefficient $G(t)$ undergoes a jump at a single point of the contour, t_1. We introduce two auxiliary functions, $(z-z_0)^{\gamma}$, $(z-t_1)^{\gamma}$, where $z_0 \in D^+$, $\gamma=\alpha+i\beta$ is some complex number. The branch points of the first function are the points z_0 and ∞, and the branch points of the second function are t_1 and ∞. In the plane cut along a line joining the points z_0 and t_1 and extending to infinity, the above functions are single valued. We form a piecewise analytic function $\omega(z)$ defined by the relations

$$\omega^+(z)=(z-t_1)^{\gamma}, \quad \omega^-(z)=\left(\frac{z-t_1}{z-z_0}\right)^{\gamma}.$$

The single-valuedness of these functions in the corresponding regions is ensured by the proper cuts. These functions are continuous everywhere on the contour L, with the exception of the point t_1.

We introduce a new function

$$\Omega(t)=\frac{\omega^-(t)}{\omega^+(t)}=(t-z_0),$$

from which

$$\frac{\Omega(t_1-0)}{\Omega(t_1+0)}=e^{-2\pi i\gamma},$$

Let us investigate the behaviour of the function $\omega(z)$ in the neighbourhood of the point t_1 and introduce a local polar co-ordinate system centred at this point:

$$\omega^+(z)=(z-t_1)^{\gamma}=e^{\gamma\ln(z-t_1)}=r^{\alpha}e^{-\beta\theta}e^{i(\beta\ln r+\alpha\theta)},$$
$$z-t_1=re^{i\theta}.$$

Consequently, when $\alpha>0$, the function $\omega^+(z)$ has a zero of order α at the point t_1; when $\alpha<0$, a pole of order $-\alpha$ (the condition for integrability leads to the condition $-1<\alpha$). Finally, when $\alpha=0$, the function $\omega^+(z)$ remains bounded, but does not tend to a definite limit as the point z tends to the point t_1. The function $\omega^-(z)$ has similar properties in the neighbourhood of the point t_1.

We now turn to the direct solution of the homogeneous Riemann problem (3.4).It will be recalled that for the present we are considering the case when the function $G(t)$ has just one point of discontinuity. We now define γ as follows:

$$\gamma=\frac{1}{2\pi i}\ln\frac{G(t_1-0)}{G(t_1+0)} \tag{5.2}$$

We next form the functions $\omega^+(z)$ and $\omega^-(z)$ corresponding to the given value of γ. Let us introduce a new piecewise analytic function,

$\Phi_1(z)$, setting $\Phi(z) = \omega(z)\,\Phi_1(z)$. The boundary condition (3.4) then becomes

$$\Phi_1^+(t) = G_1(t)\,\Phi_1^-(t), \qquad G_1(t) = \Omega(t)\,G(t). \tag{5.3}$$

The coefficient $G_1(t)$ of the auxiliary Riemann problem is now a continuous function on the entire contour L, including the point t_1. Since the piecewise analytic function $\Phi_1(z)$ is continuous, the singularities of the function $\Phi(z)$ are determined only by the behaviour of the function $\omega(z)$ near the point of discontinuity. The latter is governed by formula (5.2) and depends only on the choice of the branch of the logarithm. If only bounded solutions are permitted, the inequality $0 \leqslant \operatorname{Re}\gamma < 1$ must be fulfilled. If unbounded solutions are allowed, the inequality $-1 \leqslant \operatorname{Re}\gamma < 0$ must hold. In the latter inequality the left bound is associated with the integrability condition.

Of some interest is the construction of similar estimates directly for the coefficient $G(t)$. Denote by θ the increment of either branch of the argument of $G(t)$ on passing once round the contour L. It is obvious that θ is the jump in the argument of $G(t)$ at the point of discontinuity; hence,

$$\frac{G(t_1-0)}{G(t_1+0)} = \rho e^{i\theta},$$

and so

$$\gamma = \frac{1}{2\pi i}\ln\frac{G(t_1-0)}{G(t_1+0)} = \frac{\theta}{2\pi} - \varkappa - i\,\frac{\ln\rho}{2\pi},$$

where the integer $\varkappa$ must be chosen in accordance with the foregoing inequalities, which now take the form

$$0 \leqslant \frac{\theta}{2\pi} - \varkappa < 1, \qquad -1 < \frac{\theta}{2\pi} - \varkappa < 0.$$

Thus, for the class of bounded solutions $\varkappa = [\theta/2\pi]$,* and for the class of unbounded solutions $\varkappa = [\theta/2\pi] + 1$. If $\theta/2\pi$ is an integer, only the first condition can be fulfilled because of the restrictions on $\operatorname{Re}\gamma$. The solution, remaining bounded in the neighbourhood of the point t_1, does not tend to any limit as t_1 is approached. The point of discontinuity in this case is termed the point of automatic boundedness .

Let us now calculate the index of the auxiliary Riemann problem, i.e., the index of the function $G_1(t)$:

$$\operatorname{Ind} G_1(t) = \frac{1}{2\pi i}\ln\left[\frac{G(t_1-0)}{G(t_1+0)}\,e^{-2\pi i\varkappa}\right] = \frac{1}{2\pi i}\ln e^{2\pi i\varkappa} = \varkappa.$$

This quantity will be called the index of the original Riemann problem. Thus, when there is a discontinuity in the coefficient $G(t)$, the value of the index depends on the chosen class of solutions.

* The symbol [ ] denotes the integral part of a number.

We now extend the results obtained to the case when the coefficient $G(t)$ has discontinuities of the first kind at a set of points, $t_1, t_2, \ldots, t_n$. We can arbitrarily assign a branch for the function $\ln G(t)$ on each of the arcs (without allowing the transition from one branch to the other at the interior points of these arcs). Similarly to the foregoing we introduce a change θ_k in argument of the function $G(t)$ at each point t_k. We obtain

$$\frac{G(t_k-0)}{G(t_k+0)}=\rho_k e^{i\theta_k} \quad (k=1,\ 2,\ \ldots,\ n).$$

Let us now assume

$$\gamma_k=\frac{1}{2\pi i}\ln\frac{G(t_k-0)}{G(t_k+0)}=\frac{\theta_k}{2\pi}-\varkappa_k-i\,\frac{\ln\rho_k}{2\pi},$$

where the numbers $\varkappa_k$ are determined as before, depending on the singularity permitted at the corresponding point. We introduce a new piecewise analytic function

$$\Phi(z)=\prod_{k=1}^{n}\omega_k(z)\,\Phi_1(z),$$

where the subscript k on the function $\omega(z)$ indicates that this function is determined by the point t_k. Naturally, to each function $\omega_k(z)$ there corresponds its particular system of cuts (with the same choice of the point z_0).

For the function $\Phi_1(z)$, we obtain a boundary value problem with a continuous coefficient:

$$\Phi_1^+(t)=\prod_{k=1}^{n}(t-z_0)^{-\gamma_k}G(t)\,\Phi_1^-(t).$$

Similarly to the foregoing it can be shown that the index of the auxiliary Riemann problem, which will also be called the index of the original problem, is determined as

$$\varkappa=\varkappa_1+\varkappa_2+\ldots+\varkappa_n.$$

The construction of the final solution presents no difficulties.

We now turn to the consideration of the non-homogeneous Riemann problem (3.2). Suppose, for the present, that the function $g(t)$ satisfies the H-L condition. In conformity with the foregoing, we proceed to the solution of an auxiliary boundary value problem with a continuous coefficient:

$$\Phi_1^+(t)=\prod_{k=1}^{n}(t-z_0)^{-\gamma_k}G(t)\,\Phi_1^-(t)+\prod_{k=1}^{n}(t-t_k)^{-\gamma_k}g(t).$$

By replacing the coefficient $\prod\limits_{k=1}^{n}(t-z_0)^{-\gamma_k}G(t)$ by the quotient of the canonical functions (3.11), we arrive at a boundary condition

in the form

$$\frac{\Phi_1^+(t)}{X_1^+(t)}-\frac{\Phi_1^-(t)}{X_1^-(t)}=\frac{\prod_{k=1}^{n}(t-t_k)^{-\gamma_k}g(t)}{X_1^+(t)}.$$

The solution of this problem is of the form of (3.17) and is written as

$$\frac{\Phi_1(z)}{X_1(z)}=\Psi(z)+P_{\varkappa-1}(z), \tag{5.4}$$

where $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa-1$ when $\varkappa\geqslant 1$; when $\varkappa<1$, the polynomial is absent.

Below are the expressions for the unknown function:

$$\begin{aligned}\Phi^+(z)&=\prod_{k=1}^{n}(z-t_k)^{\gamma_k}X_1^+(z)\,[\Psi^+(z)+P_{\varkappa-1}(z)],\\ \Phi^-(z)&=\prod_{k=1}^{n}\left(\frac{z-t_k}{z-z_0}\right)^{\gamma_k}X_1^-(z)\,[\Psi^-(z)+P_{\varkappa-1}(z)].\end{aligned} \tag{5.5}$$

Here we have used the notation

$$X_1^+(z)=e^{\Gamma^+(z)},\quad X_1^-(z)=(z-z_0)^{-\varkappa}e^{\Gamma^-}(z),$$

$$\Gamma(z)=\frac{1}{2\pi i}\int_L\frac{\ln\left[(\tau-z_0)^{-\varkappa}\prod_{k=1}^{n}(\tau-z_0)^{-\gamma_k}G(\tau)\right]}{\tau-z}\,d\tau,$$

$$\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\prod_{k=1}^{n}(\tau-t_k)^{-\gamma_k}g(\tau)}{X_1^+(\tau)(\tau-z)}\,d\tau.$$

The necessity of the restriction $\operatorname{Re}\gamma_k<1$ introduced above (without proper justification) follows from the structure of the expression for the function $\Psi(z)$.

When $\varkappa<0$, the boundary value problem is solvable if conditions similar to (3.18) are fulfilled:

$$\int_L\frac{\prod_{k=1}^{n}(\tau-t_k)^{-\gamma_k}g(\tau)}{X_1^+(\tau)}\tau^{j-1}d\tau=0\quad (j=1,2,\ldots,-\varkappa-1).$$

Suppose that the function $g(t)$ has the following representation a points t_1', t_2', ..., t_r':

$$g(t)=\frac{g^*(t)}{(t-t_k')^{\gamma_k'}},$$

Here $\gamma'_k = \alpha'_k + i\beta'_k$ $(0 \leqslant \alpha'_k < 1)$, the function $g^*(t)$ satisfies the H-L condition everywhere except at points $t''_1, t''_2, \ldots, t''_s$ where it has discontinuities of the first kind.

We first assume that the points $t''_1, t''_2, \ldots, t''_s$ are different from the points of discontinuity in the coefficient $G(t)$. In this case solution (5.4) remains valid. True, by virtue of formulas (2.31) the function $\Phi^+(z)$ has a singularity of the form $(z - t'_k)^{-\gamma'_k}$ in the neighbourhood of the points t'_k, and a logarithmic singularity in the neighbourhood of the points t''_k. The coincidence of any one of the points t''_k with a point of discontinuity in the coefficient $G(t)$ introduces no change in the behaviour of the function

$$\frac{\prod_{k=1}^{n} (\tau - t_k)^{-\gamma_k} g(\tau)}{X_1^+(\tau)}.$$

The coincidence of any one of the points t'_k with a point t_k may lead to adding up of singularities. If $\alpha_k + \alpha'_k < 1$, the theory considered is applicable only for an unbounded solution at the point t_k (excluding $\alpha'_k = 0$).

By way of illustration we shall study the Riemann problem for a system of unclosed contours $L_1, L_2, \ldots, L_n$ (whose set will be further denoted by L) when the coefficient $G(t)$ is identically equal to a certain constant $c < 0$. This case has important applications in the theory of elasticity (see Secs. 26 and 27). The ends of the arcs L_k are denoted by a_k and b_k (a_k is the beginning of description, b_k is the end). In conformity with the foregoing, we construct a sufficiently arbitrary system of arcs L'_k joining the ends b_k and a_{k+1} (this system of arcs will be denoted by L'). The set of contours L and L' forms a closed contour.

Thus, we again arrive at the Riemann boundary value problem (3.2); the coefficient $G(t) = c < 0$ on the arcs L_k, and $G(t) = 1$ on the arcs L'_k. The coefficient $g(t)$ is zero on the arcs L'_k. Let us determine, at all points a_k and b_k, the quotients*

$$\frac{G(a_k - 0)}{G(a_k + 0)} = \frac{1}{c} e^{-i\pi}, \quad \gamma_k = -\frac{1}{2} - \varkappa_k + i\beta, \quad \frac{G(b_k - 0)}{G(b_k + 0)} = ce^{i\pi},$$

$$\gamma'_k = \frac{1}{2} - \varkappa'_k - i\beta, \quad \beta = \frac{\ln|c|}{2\pi}.$$

The absence or presence of a prime in the expressions for γ_k indicates that the quantity is taken, respectively, at the ends a_k or b_k.

We first seek a solution that becomes infinite at all ends; hence, the integers $\varkappa_k$ and $\varkappa'_k$ defined by the preceding formulas are respec-

* The choice of values of θ introduced here is somewhat different from that suggested previously (see p. 66), which is reflected in another choice of numbers $\varkappa$ (see F. D. Gakhov [1]).

tively equal to $\varkappa = 0$ and $\varkappa' = 1$, from which $\gamma_k = -\frac{1}{2} + i\beta$, $\gamma_k' = -\frac{1}{2} - i\beta$. Thus, the index of the Riemann problem is found to be equal to the number of arcs n, and the final expression for the coefficient of the auxiliary Riemann problem is of a very simple form, namely $G_1(t) = (t - z_0)^n$. The canonical function is found in an elementary way: $X_1^+(z) = 1$, $X_1^-(z) = (z - z_0)^{-n}$.

Thus, the final solution following from formulas (5.5) is represented as

$$\Phi(z) = \frac{\prod_{k=1}^{n} \left(\frac{z-b_k}{z-a_k}\right)^{i\beta}}{\prod_{k=1}^{n} \sqrt{(z-a_k)(z-b_k)}} \int_L \frac{\prod_{k=1}^{n} \sqrt{(\tau-a_k)(\tau-b_k)}\, g(\tau)}{\prod_{k=1}^{n} \left(\frac{\tau-b_k}{\tau-a_k}\right)^{i\beta} (\tau-z)} d\tau + \frac{\prod_{k=1}^{n} \left(\frac{z-b_k}{z-a_k}\right)^{i\beta}}{\prod_{k=1}^{n} \sqrt{(z-a_k)(z-b_k)}} P_{n-1}(z), \tag{5.6}$$

where $P_{n-1}(z)$ is a polynomial of degree $n - 1$. It should be emphasized that the radicals appearing in the integrand are not multiple-valued functions since the chosen system of cuts ($z_0 - a_k - \infty$, $z_0 - b_k - \infty$) introduces single-valuedness into their determination. In other words, by the radical must be understood either branch of the multiple-valued function in the plane cut along the arcs L_k.

We now proceed to a solution bounded at all points a_k, but unbounded, as before, at the ends b_k. In this case we have the equalities $\varkappa_k = -1$ and $\gamma_k = \frac{1}{2} + i\beta$. The index of the Riemann problem is zero, and the solution is, by (5.5),

$$\Phi(z) = \frac{\prod_{k=1}^{n} \left(\frac{z-b_k}{z-a_k}\right)^{i\beta} \prod_{k=1}^{n} \sqrt{(z-a_k)}}{\prod_{k=1}^{n} \sqrt{(z-b_k)}} \times \int_L \frac{\prod_{k=1}^{n} \sqrt{(\tau-b_k)}\, g(\tau)}{\prod_{k=1}^{n} \sqrt{(\tau-a_k)} \left(\frac{\tau-b_k}{\tau-a_k}\right)^{i\beta}} \frac{d\tau}{(\tau-z)}. \tag{5.7}$$

In the case when the solution being determined is bounded at all ends, the index is equal to —n, and the solution is of the form

$$\Phi(z)=\prod_{k=1}^{n}\left(\frac{z-b_k}{z-a_k}\right)^{i\beta}\sqrt{(z-a_k)(z-b_k)}\times$$
$$\times\int_L \frac{1}{\prod\limits_{k=1}^{n}\left(\frac{\tau-b_k}{\tau-a_k}\right)^{i\beta}\sqrt{(\tau-a_k)(\tau-b_k)}}\frac{g(\tau)}{\tau-z}\,d\tau. \qquad (5.8)$$

Expression (5.8) satisfies the condition at infinity (and hence is the solution of the Riemann problem) if the following relations are fulfilled:

$$\int_L \frac{g(\tau)}{\prod\limits_{k=1}^{n}\left(\frac{\tau-b_k}{\tau-a_k}\right)^{i\beta}(\tau-a_k)(\tau-b_k)}\tau^{j-1}\,d\tau=0 \quad (j=1,\ 2,\ \ldots,\ n-1). \qquad (5.9)$$

Analysis of solutions (5.6) to (5.8) shows that, according to the restrictions imposed at the ends a_k and b_k, the corresponding radicals must be transferred from the numerator to the denominator and vice versa. The index is always equal to the number of arcs minus the number of ends at which the solution is bounded.

It should be noted that the integrals appearing in solutions (5.6) to (5.8) are evaluated in closed form if the function $g(t)$ is a polynomial (see N. I. Muskhelishvili [4]).

6. Singular Integral Equations in the Case of Discontinuous Coefficients and Unclosed Contours

Just as in Sec. 4 the theory of singular integral equations is constructed on the basis of the theory of the Riemann boundary value problem for closed contours, so the corresponding theory of singular equations is constructed on the basis of the boundary value problem for unclosed contours.

The integral equation for the case under consideration is identical in appearance with Eq. (4.1) or (4.1′) provided that the integration is extended over the whole set of unclosed contours L_j ($j = 1, 2, \ldots, n$) symbolically denoted, as in Sec. 5, by L. We thus consider integral equations of the form

$$K\varphi=a(t)\,\varphi(t)+\frac{b(t)}{\pi i}\int_L\frac{\varphi(t)}{\tau-t}\,d\tau+\int_L k(t,\ \tau)\,\varphi(\tau)\,d\tau=f(t), \qquad (6.1)$$

$$a(t),\ b(t),\ f(t)\in H(A,\ \lambda).$$

By the companion equation is meant, as before, Eq. (4.2).

From an analysis of the expressions for the solutions of singular integral equations for closed contours it may be established that the unknown function belongs to the same class H-L as the right-hand side. It is obvious that in the case under consideration this result no longer holds. Moreover, in constructing the solution it is necessary to preassign, from some additional (possibly physical) considerations, the order of singularity at the end points (as in considering the boundary value problem).

It is convenient for further considerations to introduce a new indexing of the ends of the contours L_k. Denote them all by the same letter c with a subscript such that the solution is bounded at the ends c_k $(k = 1, 2, \ldots, q)$ and unbounded, but, of course, integrable at the remaining ends c_k $(k = q + 1, q + 2, \ldots, 2n)$. The solution of a singular integral equation will be said to belong to the class $h(c_1, c_2, \ldots, c_q)$ if it is bounded at the points c_k $(k = 1, 2, \ldots, q)$ and unbounded at the points c_k $(k = q + 1, q + 2, \ldots, 2n)$. By the companion solution of the companion equation is meant the solution in the class $h(c_{q+1}, c_{q+2}, \ldots, c_{2n})$, called the companion class. As in the case of closed contours, we require the fulfilment of the condition $a^2(t) - b^2(t) \neq 0$.

We begin with the consideration of the characteristic equation

$$K^0\varphi = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - t} = f(t). \tag{6.2}$$

By means of the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau$$

we pass to the auxiliary Riemann problem

$$[a(t) + b(t)]\,\Phi^+(t) = [a(t) - b(t)]\,\Phi^-(t) + f(t). \tag{6.3}$$

The general solution of this problem is written down with the aid of (5.4), where by $X(t)$ is meant a canonical function bounded at the same points $c_1, c_2, \ldots, c_q$, or, in the terminology introduced, a function of the class $h(c_1, c_2, \ldots, c_q)$. The solution of Eq. (6.1) is then represented as

$$\varphi(t) = a(t)f(t) - \frac{b(t)Z(t)}{\pi i}\int_L \frac{f(\tau)}{Z(\tau)}\frac{d\tau}{\tau - t} + b(t)Z(t)P_{\varkappa-1}(t), \tag{6.4}$$

where

$$Z(t) = [a(t) + b(t)]\,X^+(t) = [a(t) - b(t)]\,X^-(t) = \prod_{k=1}^{n}(t - c_k)^{-\varkappa_k}e^{\Gamma(t)}.$$

$$\Gamma(t) = \frac{1}{2\pi i}\int \frac{\ln\{[a(t) - b(t)]/[a(t) + b(t)]\}}{\tau - t}\,d\tau, \qquad \varkappa = \sum_{k=1}^{n}\varkappa_k.$$

The constants $\varkappa_k$ are determined in accordance with the nature of singularities at the ends (p. 66), the polynomial is of degree $\varkappa - 1$ since $\Phi(\infty)$ is automatically zero. If the index $\varkappa \leqslant 0$, the polynomial is absent. When $\varkappa < 0$, the solution of a problem of the class in question exists if and only if the following conditions are fulfilled:

$$\int_L \frac{f(t)}{Z(\tau)} \tau^{j-1}\, d\tau = 0 \quad (j = 1, 2, \ldots, -\varkappa). \tag{6.5}$$

Consider now an equation companion to the characteristic one:

$$K'\psi = a(t)\psi(t) - \frac{1}{\pi i} \int_L \frac{b(\tau)\psi(\tau)}{\tau - t}\, d\tau = f_1(t). \tag{6.6}$$

By means of the Cauchy-type integral

$$\Omega(t) = \frac{1}{2\pi i} \int_L \frac{b(\tau)\psi(\tau)}{\tau - z}\, d\tau$$

we pass to the boundary value problem

$$[a(t) - b(t)]\,\Omega^+(t) = [a(t) + b(t)]\,\Omega^-(t) + f_1(t). \tag{6.7}$$

From the foregoing it follows immediately that the function $X'(z) = 1/X(z)$ is the canonical function of problem (6.7) in the companion class. Consequently, the index of the companion problem in the companion class is equal to the negative of the index of the basic problem.

The general solution of Eq. (6.6) in the class $h(c_{q+1}, c_{q+2}, \ldots, c_{2n})$ is written as

$$\psi(t) = a(t) f_1(t) + \frac{1}{\pi i Z(t)} \int_L \frac{Z(\tau)\, b(\tau) f_1(\tau)}{\tau - t}\, d\tau + \frac{1}{Z(t)} P_{-\varkappa-1}(t). \tag{6.8}$$

When $-\varkappa \leqslant 0$, the polynomial is absent; when $-\varkappa < 0$, the solvability requires the fulfilment of the conditions

$$\int_L Z(\tau)\, b(\tau) f_1(\tau)\, \tau^{j-1}\, d\tau = 0 \quad (j = 1, 2, \ldots, \varkappa). \tag{6.9}$$

It follows from representation (6.8) that the solution of the homogeneous companion equation is

$$\psi_j(t) = \frac{1}{Z(t)} t^{j-1} \quad (j = 1, 2, \ldots, -\varkappa).$$

The conditions for the solvability of the original equation (6.1), previously given as (6.5), may therefore be represented in conventional form:

$$\int_L f(\tau)\, \psi_j(\tau)\, d\tau = 0 \quad (j = 1, 2, \ldots, -\varkappa). \tag{6.5'}$$

On comparing the results obtained, we arrive at the same formulation of Noether's theorems for characteristic singular equations on unclosed contours as for closed contours (see Sec. 4). For the solvability of the integral equation (6.3) in the class h $(c_1, c_2, \ldots, c_q)$ it is necessary and sufficient that conditions (6.5′) should be fulfilled, and for the solvability of the companion equation (6.4) in the companion class it is necessary that conditions (6.9) should be fulfilled; these conditions may also be represented as

$$\int_L f_1(\tau)\,\varphi_j(\tau)\,d\tau = 0 \quad (j = 1, 2, \ldots, \varkappa), \tag{6.9′}$$

where $\varphi_j(t)$ is the solution of the homogeneous original equation.

Note that the index of the equation is also equal to the difference in the number of solutions of the homogeneous original (k) and companion (k') equations. Indeed, $k = \varkappa$, $k' = 0$ when $\varkappa \geqslant 0$, and $k = 0$, $k' = -\varkappa$ when $\varkappa < 0$.

We now turn to the complete singular equation (6.1). Naturally, the method of studying singular integral equations on closed contours described in Sec. 4 is extended to the case under investigation with appropriate complications due to the presence of singularities in the kernels and in the solution being sought. We agree to consider the index of the characteristic part as the index of the complete equation [of course, in the same class h $(c_1, c_2, \ldots, c_q)$].

In the case when $\varkappa \geqslant 0$ we perform left-hand regularization by means of operator (6.6), as a result of which we arrive at an integral equation

$$K'K\varphi = K'f, \tag{6.10}$$

equivalent to the original one. It should be noted that this equation is not strictly a Fredholm one since its kernel has singularities at the ends of the contours. It should be recalled, however, that in Sec. 1 we have proved the fulfilment of the Fredholm alternatives for this case. The presence of singularities for the unknown function has been considered (see Sec.5) in reference to boundary value problems. It is essential that the companion classes are introduced so as to avoid adding up of singularities (which could lead to the appearance of non-integrable singularities).

In the case of $\varkappa < 0$ we must perform right-hand regularization. Note that when two functions, $\varphi(t)$ and $\psi(t)$, belong to the companion classes, identity (4.3) holds for unclosed contours. The proofs of Noether's theorems themselves reproduce the proofs given for the case of closed contours with due regard for the specific structure of operator (6.6) and the circumstance that the solutions of the original and companion equations are considered in the companion classes.

7. Two-dimensional Singular Integrals

In presenting the theory of Fredholm integral equations, the case of one variable has been considered only for shortness of writing, while the same methods can be used to obtain completely similar results in the case of arbitrary dimension. The extension of the foregoing theory of one-dimensional singular integrals and integral equations to the case of a larger number of dimensions is, for the most part, impossible. The construction of the corresponding theory involves the development of special methods. Note also that the case of two dimensions can be studied by simpler means than the case of arbitrary dimension. In the following discussion we shall therefore restrict ourselves to this case bearing in mind, of course, that the integral equations for three-dimensional problems of the theory of elasticity (see Chap. VI) are two-dimensional equations. It should be noted, however, that the theory of two-dimensional equations is in great part extended to the general case.

Let a function $F(q)$ be defined on a certain surface S (considered to be a plane to begin with, and denoted by Π). Consider an arbitrary point q_0 and isolate, on Π, a part Π_ε at a distance less than a certain ε from the point q_0. Suppose that the function $F(q)$ is summable in the remainder of the plane for any ε. If the limit

$$\lim_{\varepsilon\to 0}\int\limits_{\Pi-\Pi_\varepsilon} F(q)\,dS_q,$$

exists, it is called a *singular integral* and denoted by

$$\int\limits_{\Pi} F(q)\,dS_q.$$

We shall further consider singular integrals of the form

$$\int\limits_{\Pi} K(q_0, q)\,u(q)\,dS_q. \tag{7.1}$$

We first assume that the function $u(q)$ satisfies the H-L condition and decreases at infinity as $1/|q|^\beta$ $(\beta > 0)$. Our study is restricted to the case when the kernel $K(q_0, q)$ admits the representation

$$K(q_0, q) = \frac{1}{r^2(q_0, q)} f(q_0, \theta), \tag{7.2}$$

where $r = r(q_0, q)$ is the distance between the points q_0 and q, and θ is the angle that a ray going from the point q_0 to the point q makes with a fixed direction. The function $f(q_0, \theta)$ is called the *characteristic of a singular integral* of this kind, and the function $u(q)$ is, as before, the density function.

Let us establish conditions for the existence of the singular integrals introduced above. We have

$$\int_{\Pi} K(q_0, q)\, u(q)\, dS_q = \int_{r>1} K(q_0, q)\, u(q)\, dS_q + \\ + \int_{r<1} K(q_0, q)\, [u(q) - u(q_0)]\, dS_q + u(q_0) \int_{r<1} K(q_0, q)\, dS_q. \quad (7.3)$$

In view of the restrictions introduced the first two integrals are absolutely convergnt. To evaluate the third integral, we introduce a local (in the neighbourhood of the point q_0) polar co-ordinate system. We then have

$$\int_{r<1} K(q_0, q)\, dS_q = \lim_{\varepsilon \to 0} \int_{\varepsilon<r<1} K(q_0, q)\, dS_q = \lim \ln \frac{1}{\varepsilon} \int_{L} f(q_0, \theta)\, dL,$$

where L is a circle of radius ε centred at the point q_0. The limit of the right-hand side exists if and only if

$$\int_{L} f(q_0, \theta)\, dL = 0. \quad (7.4)$$

Thus, (7.4) is a condition for the existence of singular integrals of the class considered here. In the following discussion it will be assumed that this condition is always fulfilled.

In view of the foregoing and from (7.3) we obtain the following representation for singular integrals:

$$\int_{\Pi} K(q_0, q)\, u(q)\, dS_q = \int_{r<\delta} K(q_0, q)\, u(q)\, dS_q + \\ + \int_{r<\delta} K(q_0, q)\, [u(q) - u(q_0)]\, dS_q. \quad (7.5)$$

The possibility of replacing unity (in determining the region of integration) by an arbitrary constant δ is obvious.

Suppose that for some reason or other (see the case of an arbitrary surface in later parts of the book) the shape of the region being cut out is different from a circle. Let the equation of the boundary of this region σ_ε (previously a circle) be $r = \alpha(\varepsilon, q_0, \theta)$. Suppose that the following limit exists:

$$\lim_{\varepsilon \to 0} \frac{\alpha(\varepsilon, q_0, \theta)}{\varepsilon} = \beta(q_0, 0) > 0.$$

We now extend representation (7.5):

$$\lim_{\varepsilon\to 0}\int\limits_{\Pi-\sigma_\varepsilon}\frac{f(q_0,\theta)}{r^2}u(q)\,dS_q=\int\limits_{r>\delta}\frac{f(q_0,\theta)}{r^2}u(q)\,dS_q+$$
$$+\lim_{\varepsilon\to 0}\int\limits_{\alpha<r<\delta}\frac{f(q_0,\theta)}{r^2}[u(q)-u(q_0)]\,dS_q-$$
$$-u(q_0)\lim_{\varepsilon\to 0}\int\limits_L f(q_0,\theta)\ln\alpha(\varepsilon,q_0,\theta)\,dL.$$

From condition (7.4) we obtain

$$\int\limits_L f(q_0,\theta)\ln\alpha(\varepsilon,q_0,\theta)\,dL=\int\limits_L f(q_0,\theta)\ln\frac{\alpha(\varepsilon,q_0,\theta)}{\varepsilon}\,dL.$$

Consequently,

$$\lim_{\varepsilon\to 0}\int\limits_{\Pi-\sigma_\varepsilon}\frac{f(q_0,\theta)}{r^2}u(q)\,dS_q=$$
$$=\int\limits_{\Pi}\frac{f(q_0,\theta)}{r^2}u(q)\,dS_q-u(q_0)\int\limits_L f(q_0,\theta)\ln\beta(q_0,\theta)\,dL. \qquad (7.6)$$

Thus, it has been shown that a singular integral can be defined in different ways [with an arbitrary function $\beta(q_0,\theta)$], but the condition for its existence is always equality (7.4). Naturally, when $\beta=1$, formulas (7.5) and (7.6) lead to the same result.

Let us prove that the singular integral

$$v(q_0)=\int\limits_{\Pi}K(q_0,q)\,u(q)\,dS_q$$

satisfies the H-L condition if the characteristic $f(q_0,\theta)$ is continuously differentiable in the Cartesian co-ordinates of the point q_0 and the angle θ (consequently, we have the estimate grad $K(q_0,q)=$ $=O(r^{-3})$ when $r\to 0$).* We proceed from representation (7.5). The first term is a function having continuous derivatives. We shall therefore consider only the second term, which, for convenience, is denoted by $\omega(q_0)$. Let the constant $|h|<\delta/2$; then

$$\omega(q_0+h)-\omega(q_0)=\int\limits_{|q_0+h-q|<\delta}K(q_0+h,q)[u(q)-u(q_0+h)]\,dS_q-$$
$$-\int\limits_{|q_0-q|<\delta}K(q_0,q)[u(q)-u(q_0)]\,dS_q=$$

* The result thus formulated is known as Giraud's theorem (see S. G. Mikhlin [5]), which may be regarded as a generalization of the Plemelj-Privalov theorem (Sec. 2).

$$= \int\limits_{|q_0-q|<\delta-|h|} K(q_0+h,\ q)[u(q)-u(q_0+h)]\,dS_q -$$

$$- \int\limits_{|q_0-q|<\delta-|h|} K(q_0,\ q)[u(q)-u(q_0)]\,dS_q +$$

$$+ \int\limits_{(|q_0+h-q|<\delta)\cap(|q_0-q|>\delta-|h|)} K(q_0+h,\ q)[u(q)-u(q_0+h)]\,dS_q -$$

$$- \int\limits_{\delta-|h|<|q_0-q|<\delta} K(q_0,\ q)[u(q)-u(q_0)]\,dS_q. \qquad (7.7)$$

The integrands in the last two terms are bounded, and the area of the surface of integration is of order h. Each of the first two integrals is divided into integrals over the area of a circle $|q_0 - q| < 2|h|$ and a ring $2|h| < |q_0 - q| < \delta - |h|$. Taking into account the inequalities

$$|u(q)-u(q_0)|\,|K(q_0,\ q)| < Cr^{\alpha-2},$$

$$|u(q)-u(q_0+h)|\,|K(q_0+h,\ q)| < Cr_1^{\alpha-2} \quad (r_1 = |q_0+h-q|),$$

we obtain the estimates

$$\left|\int\limits_{r<2|h|} K(q_0,\ q)[u(q)-u(q_0)]\,dS_q\right| < C \int\limits_{r<2|h|} r^{\alpha-2}\,dS_q = C_1|h|^{\alpha}, \quad (7.8)$$

$$\left|\int\limits_{r<2|h|} K(q_0+h,\ q)[u(q)-u(q_0+h)]\,dS_q\right| <$$

$$< C \int\limits_{r<2|h|} r_1^{\alpha-2}\,dS_q < C \int\limits_{r_1<3|h|} r_1^{\alpha-2}\,dS_q = C_2|h|^{\alpha}.$$

To evaluate the surface integrals, we use the identity transformation

$$[u(q)-u(q_0+h)]K(q_0+h,\ q)-[u(q)-u(q_0)]K(q_0,\ q) =$$
$$= [u(q)-u(q_0+h)][K(q_0+h,\ q)-K(q_0,\ q)] -$$
$$-[u(q_0+h)-u(q)]K(q_0,\ q).$$

By condition (7.3), the integral of the second term vanishes. From the conditions of the theorem we have the estimate

$$|K(q_0+h,\ y)-K(q_0,\ y)| < \frac{C_3}{|q_0'-q|^3},$$

where q_0' is a point situated between q_0 and $q_0 + h$. Since $|q_0' - q| > r - |h|$ and $r \geqslant 2h$ in the region of integration, it follows that $|q_0' - q| > r/2$; since the function $u(q)$ satisfies the H-L con-

dition with index α, we have

$$|u(q)-u(q+h)||K(q_0+h,\ q)-K(q_0,\ q)|<$$

$$<\frac{C_4|h||q_0+h-q|^\alpha}{r^3}\leqslant\frac{C_4|h|(r+|h|)^\alpha}{r^3}.$$

Taking into account these inequalities and Hölder's inequality $(r+|h|)^\alpha<r^\alpha+|h|^\alpha$ (see S. L. Sobolev [2]), we finally obtain

$$|u(q)-u(q+h)||K(q_0+h,\ q)-K(q_0,\ q)|<C_4\frac{|h|}{r^{3-\alpha}}+C_4\frac{|h|^{\alpha+1}}{r^3}.$$

By using the preceding inequality, we arrive at the required estimate for the integral over the area of the ring:

$$\left|\int\limits_{2|h|<|q_0-q|<\delta-|h|}[K(q_0+h,\ q)-K(q_0,\ q)][u(q)-u(q_0+h)]\,dS_q\right|<$$

$$<C_5|h|\int\limits_{2|h|}^{\delta-|h|}\frac{dr}{r^{2-\alpha}}+C_6|h|^{\alpha+1}\int\limits_{2|h|}^{\delta-|h|}\frac{dr}{r^2}<C_7|h|^\alpha.$$

Giraud's theorem may thus be considered proved.

Let us discuss one auxiliary question. Under certain conditions (see S. L. Sobolev [3]) the derivative of an improper integral with respect to a parameter is represented as the integral of the derivative of the kernel. Below is given a special case when the resulting integral is singular.

Consider the improper integral

$$v(q_0)=\int\limits_{\Pi}\frac{f(q_0,\ \theta)}{r}u(q)\,dS_q.$$

Assume that the function $u(q)$ satisfies the conditions stated above, and the function $f(q_0,\ \theta)$ has derivatives with respect to the Cartesian co-ordinates of the point q_0 and the angle θ satisfying the H-L condition. We have

$$\frac{\partial v}{\partial x_k^0}=\frac{\partial}{\partial x_k^0}\lim_{\varepsilon\to 0}\int\limits_{r>\varepsilon}\frac{f(q_0,\ \theta)}{r}u(q)\,dS_q\quad(k=1,\ 2)\tag{7.9}$$

(x_1^0, x_2^0 and x_1, x_2 are the Cartesian co-ordinates of the points q_0 and q, respectively). We have the identity

$$\frac{\partial}{\partial x_k^0}\int\limits_{r>\varepsilon}\frac{f(q_0,\ \theta)}{r}u(q)\,dS_q=$$

$$=\int\limits_{r>\varepsilon}\frac{\partial}{\partial x_k^0}\left[\frac{f(q_0,\ \theta)}{r}\right]u(q)\,dS_q-\int\limits_{r=\varepsilon}\frac{f(q_0,\ \theta)}{r}\frac{\partial r}{\partial x_k^0}u(q)\,dL.$$

Both integrals on the right converge uniformly to their limits as $\varepsilon \to 0$. We can therefore interchange the order of differentiation and limiting process in (7.9). We obtain the required equality

$$\frac{\partial}{\partial x_k^0} \int\limits_{\Pi} \frac{f(q_0, \theta)}{r} u(q)\, dS_q = \int\limits_{\Pi} \frac{\partial}{\partial x_k^0} \left[\frac{f(q_0, \theta)}{r}\right] u(q)\, dS_q -$$

$$- u(q_0) \int\limits_{r=\varepsilon} f(q_0, \theta) \frac{\partial r}{\partial x_k}\, dL \quad \left(\cos(r, x_k) = \frac{\partial r}{\partial x_k}\right). \tag{7.10}$$

As in the theory of one-dimensional singular equations, the question of the composition of two singular integrals plays an important part in the theory of two-dimensional (and, in general, multidimensional) equations. Naturally, it must be preceded by the consideration of the composition of a singular and a regular integral.

Let $v(q)$ be, as before, a singular integral:

$$v(q_0) = \int\limits_{\Pi} \frac{f(q_0 \theta)}{r^2} u(q_1)\, dS_{q_1},$$

and let $w(q)$ be a regular integral:

$$w(q_0) = \int\limits_{\Pi} \frac{f_1(q_0, q)}{r^\gamma} u(q)\, dS_q,$$

where $f_1(q_0, q)$ is a bounded function, $\gamma < 2$.

We proceed to the iterated integral

$$w(q_0) = \int\limits_{\Pi} \frac{f(q_0, q)}{r^\gamma (q_0, z)}\, dS_q \int\limits_{\Pi} \frac{f(q, 0)}{r^2(q, q_1)} u(q_1)\, dS_{q_1}. \tag{7.11}$$

Let $r(q_0, q)$ be denoted by r, and $r(q, q_1)$ by r_1.

We prove that the order of integration in expression (7.11) can be interchanged, as a result of which we arrive at a regular representation of the function $w(q)$ with the use of $u(q)$:

$$w(q_0) = \int\limits_{\Pi} \frac{f_1(q_0, q)}{r^\gamma}\, dS_q \lim_{\varepsilon\to 0} \int\limits_{r_1>\varepsilon} \frac{f(q, 0)}{r_1^2} u(q_1)\, dS_{q_1} =$$

$$= \lim_{\varepsilon\to 0} \int\limits_{\Pi} \frac{f_1(q_0, q)}{r^\gamma}\, dS_q \int\limits_{r_1>\varepsilon} \frac{f(q_1, \theta)}{r_1^2} u(q_1)\, dS_{q_1} =$$

$$= \lim_{\varepsilon\to 0} \int\limits_{\Pi} u(q_1)\, dS_{q_1} \int\limits_{r_1>\varepsilon} \frac{f_1(q_0, q) f(q, \theta)}{r^\gamma r_1^2}\, dS_q.$$

The above transformations have been possible because the singular integral on the left-hand side of the equalities tends uniformly to

its limit. Consider the inner integral:

$$Q(q_0, q_1) = \lim_{\varepsilon\to 0} \int_{r_1>\varepsilon} \frac{f_1(q_0, q) f(q, \theta)}{r^\gamma r_1^2} dS_q =$$

$$= \int_\Pi \frac{f_1(q_0, q) f(q, \theta) - f_1(q_0, q_1) f(q_1, \theta)}{r^\gamma r_1^2} dS_q +$$

$$+ f_1(q_0, q_1) \lim_{\varepsilon\to 0} \int_{r_1>\varepsilon} \frac{f(q_1, \theta)}{r^\gamma r_1^2} dS_q. \quad (7.12)$$

Note that the first integral is regular. Let us represent the last integral as an iterated integral in polar co-ordinates:

$$\int_L f(q_1, \theta)\, dL \int_\varepsilon^\infty \frac{dr_1}{r^\gamma r_1}.$$

We make a change of variables in the inner integral, namely $r_1 = r_2 t$, where r_2 is the distance from q_0 to q_1. Let the angle qq_0q_1 be

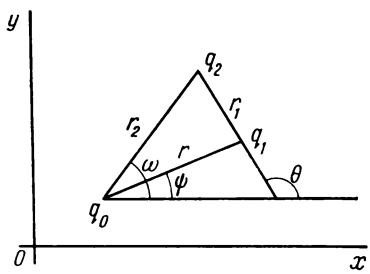

Fig.4. Schematic arrangement of points in a plane

denoted by ψ. Then

$$\int_\varepsilon^\infty \frac{dr_1}{r^\gamma r_1} = \frac{1}{r_2^\gamma} \int_{\varepsilon/r_2} \frac{dt}{t(1+t^2-2t\cos\psi)^{\gamma/2}} =$$

$$= \frac{1}{r_2^\gamma}\left[\ln\frac{r_2}{\varepsilon} + B(r_2, \psi)\right] + O(\varepsilon),$$

where $B(r_2, \psi)$ is a bounded function. Consequently, the function $Q(q_0, q_1)$ has a singularity of order $r_2^{-\gamma}$, which was to be proved.

A similar result is obtained when the composition is performed in reverse order.

Consider now two singular integrals (see the notation in Fig. 4):

$$v(q_0) = \int_\Pi K_1(q_0, q_1)\, u(q_1)\, dS_{q_1},$$
$$w(q_0) = \int_\Pi K_2(q_0, q_1)\, v(q_1)\, dS_{q_1}. \quad (7.13)$$

We shall first study a special case. Let $K_1(q_0, q) = e^{in\psi}/r^2$, $K_2(q_0, q) = \cos\psi/r^2$ and let n be an integer. Formula (7.10) allows the second integral of (7.13) to be rewritten as

$$w(q_0) = \int_{\Pi} v(q_1)\frac{\cos\psi}{r^2}\,dS_{q_1} = -\frac{\partial}{\partial x_i^0}\int_{\Pi}\frac{v(q_1)}{r}\,dS_{q_1} \quad (i = 1 \text{ or } 2).$$

Then

$$w(q_0) = -\frac{\partial}{\partial x_i^0}\int_{\Pi}\frac{1}{r}\,dS_{q_1}\lim_{\varepsilon\to 0}\int_{r_1>\varepsilon} u(q_2)\frac{e^{in\theta}}{r_1^2}\,dS_{q_2}.$$

As before, we interchange the operations of integration and limiting process:

$$w(q_0) = -\frac{\partial}{\partial x_i^0}\lim_{\varepsilon\to 0}\int_{\Pi} u(q_2)\,dS_{q_2}\int_{r_1>\varepsilon}\frac{e^{in\theta}}{rr_1^2}\,dS_{q_1}.$$

The inner integral transforms into

$$\int_{r_1>\varepsilon}\frac{e^{in\theta}}{rr_1^2}\,dS_{q_1} = \int_{-\pi}^{\pi} e^{in\theta}\left\{\frac{1}{r_2}\ln\frac{r_2}{\varepsilon} + \frac{1}{r_2}\ln\frac{1}{2}\left(1 - \frac{\varepsilon\cos(\theta-\omega)}{r_2} + \sqrt{1 - 2\frac{\varepsilon}{r_2}\cos(\theta-\omega) + \frac{\varepsilon^2}{r_2^2}}\right) - \frac{2}{r_2}\ln\left|\sin\frac{\theta-\omega}{2}\right|\right\}d\theta.$$

Substituting this expression in the formula for $w(q_0)$, and taking into account that $\varepsilon \to 0$, we arrive at the formula

$$w(q_0) = 2\frac{\partial}{\partial x_i^0}\int_{\Pi}\frac{u(q_2)}{r_2}\,dS_{q_2}\int_{-\pi}^{\pi} e^{in\theta}\ln\left|\sin\frac{\theta-\omega}{2}\right|d\theta. \qquad (7.14)$$

The inner integral in (7.14) is taken explicitly:

$$\int_{-\pi}^{\pi} e^{in\theta}\ln\left|\sin\frac{\theta-\omega}{2}\right|d\theta = \mp\frac{\pi e^{in\omega}}{n};$$

the upper sign corresponds to $n > 0$, and the lower to $n < 0$. We obtain

$$w(q_0) = \mp\frac{\partial}{\partial x_i^0}\frac{2\pi}{n}\int_{\Pi} u(q_1)\frac{e^{in\psi}}{r}\,dS_q. \qquad (7.15)$$

By performing the differentiation [according to (7.10)], we arrive at the solution of the problem of composition of the singular integrals

under consideration:

$$w(q_0) = \begin{cases} \pm 2\pi \int\limits_{\Pi} u(q_1) \frac{e^{in\psi}}{nr^2} (\cos\psi + in\sin\psi)\, dS_{q_1} & (n \neq \pm 1), \\ 2\pi \int\limits_{\Pi} u(q_1) \frac{e^{2i\psi}}{r^2}\, dS_{q_1} - 2\pi^2 u(q_0) & (n = 1), \\ 2\pi \int\limits_{\Pi} u(q_1) \frac{e^{-2i\psi}}{r^2}\, dS_{q_1} - 2\pi^2 u(q_0) & (n = -1). \end{cases} \tag{7.16}$$

For the kernels $K_1(q_0, q)$ and $K_2'(q_0, q) = i \sin\psi/r^2$, a similar discussion leads to the formulas

$$w(q_0) = \begin{cases} \pm 2\pi \int\limits_{\Pi} u(q_1) \frac{e^{in\psi}}{nr^2} (i\sin\psi + n\cos\psi)\, dS_{q_1} & (n \neq \pm 1), \\ 2\pi \int\limits_{\Pi} u(q_1) \frac{e^{2i\psi}}{r^2}\, dS_{q_1} + 2\pi^2 u(q_0) & (n = 1), \\ -2\pi \int\limits_{\Pi} u(q_1) \frac{e^{-2i\psi}}{r^2}\, dS_{q_1} - 2\pi^2 u(q_0) & (n = -1). \end{cases} \tag{7.17}$$

We introduce a symbol for a special singular operator:

$$h_n u = \frac{1}{2\pi} \int\limits_{\Pi} \frac{e^{in\psi}}{r^2} u(q_1)\, dS_{q_1},$$

the subscript 1 will be further omitted. With the notation introduced, formulas (7.16) and (7.17) may be represented (by adding and subtracting) in equivalent form:

$$\begin{aligned} hh_n u &= \pm \frac{n+1}{n} h_{n+1} u, \\ h_{-1} h_n u &= \pm \frac{1-n}{n} h_{n-1} u \qquad (n \neq \pm 1), \\ h_{-1} hu &= -u, \quad hh_{-1} u = -u. \end{aligned} \tag{7.18}$$

From the above formulas it follows that

$$h_{-1} u = h^{-1}, \quad h_2 = \frac{1}{2} h^2 u.$$

By induction, it is easy to establish the general law:

$$h_n u = \frac{1}{n} h^n u, \quad h_{-n} u = \frac{(-1)^n}{n} h^{-n} u \quad (n > 0). \tag{7.19}$$

It is convenient to introduce a new simple operator by defining it as $\tilde{h} = \frac{1}{i} h$. All preceding formulas are then transformed in an obvious

way. We give some of them needed in what follows:

$$\tilde{h}_n u = \frac{i^n}{n} \tilde{h}^n u, \quad \tilde{h}_{-n} u = \frac{i^{-n}}{n} \tilde{h}^{-n} u \quad (n>0). \tag{7.20}$$

The advisability of the above change will become evident later when the concept of the symbol of an operator is introduced. The sign tilde will be further omitted.

Consider now a singular operator:

$$\begin{gathered} Au = a_0(q_0)\, u(q_0) + \int_{\Pi} K(q_0, q)\, u(q)\, dS_1, \\ K(q_0, q) = f(q_0, \theta)/r^2. \end{gathered} \tag{7.21}$$

Assume that the coefficient $a_0(q)$ is bounded and belongs to the class H-L. We expand the characteristic $f(q_0, \theta)$ in a Fourier series:*

$$f(q_0, \theta) = \sum_{n=-\infty}^{\infty}{}' \, b_n(q_0)\, e^{in\theta} \tag{7.22}$$

Suppose that the characteristic satisfies the condition

$$\int_0^{2\pi} | f^2(q_0, \theta)\, d\theta | < C. \tag{7.23}$$

From Fourier's theory (see, for example, D. Jackson [1]) it follows that this series converges in the mean.

The foregoing enables us to represent the singular term in (7.21) as a series:

$$\int_{\Pi} u(q) \frac{f(q_0, \theta)}{r^2}\, dS_q = \sum_{n=-\infty}^{\infty}{}' \, a_n(q)\, h^n u, \tag{7.24}$$

$$a_n(q) = \frac{2\pi b_n(q)\, i^n}{n}, \quad a_{-n}(q) = (-1)^n \frac{2\pi b_{-n}(q)\, i^{-n}}{n}, \quad n>0.$$

If the term outside the integral is incorporated in this series, we obtain a representation for the operator A in symbolic form:

$$Au = \sum_{n=-\infty}^{\infty} a_n h^n u. \tag{7.25}$$

Let two singular operators, A_1 and A_2, be given, which are written down as series in the above form:

$$A_1 u = \sum_{n=-\infty}^{\infty} a_n^1 h^n u, \quad A_2 u = \sum_{n=-\infty}^{\infty} a_n^2 h^n u.$$

* The absence of a zero term is due to (7.4).

We return to the question of the composition of two such operators.* Assume that $Q(q)$ is a bounded function belonging to the class H-L. It can be proved that

$$h^n(Qu) = Qh^n u + Tu, \tag{7.26}$$

where the operator T is regular. From the structure of series (7.25) (noting the possibility, as proved above, of interchanging regular and singular operators) we have the required formula for the composition of the operators A_1 and A_2:

$$A_2 A_1 u = \sum_{k=-\infty}^{\infty}{}' \left\{ \sum_{n=-\infty}^{\infty} a^1_{k-n}(q)\, a^2_n(q) \right\} h^k u + Tu, \tag{7.27}$$

where T is a regular operator. Note that the presence of regular terms in the operators A_1 and A_2 entails only an appropriate modification of the operator T.

We introduce the concept of the symbol** of the singular operator A (7.21). The *symbol* is a complex-valued function $\Phi(q_0, \lambda)$ defined by the series

$$\Phi(q_0, \lambda) = \sum_{n=-\infty}^{\infty} a_n(q_0)\, e^{in\lambda} \quad (-\pi < \lambda < \pi). \tag{7.28}$$

The concept of a symbol can be extended to operators of a more general form, namely $A' = A + T$ (T is a completely continuous operator). The symbol of the operator A' is defined as the symbol of the operator A. Obviously, if such a definition is adopted, the symbol of a singular operator is independent of regular terms, and in the following discussion, for convenience in giving formulations, they will be ascribed a symbol equal to zero. Thus, to the sum of singular operators corresponds the sum of the symbols. It can easily be verified that to their composition corresponds the product of the symbols.

Note that the Fourier coefficients of the characteristic are identical in modulus with the Fourier coefficients of the derivative $\partial\Phi/\partial\lambda$, and in consequence we have the identity

$$\int_{-\pi}^{\pi} |f^2(q_0, \theta)|\, d\theta = \int_{-\pi}^{\pi} \left|\frac{\partial\Phi}{\partial\lambda}\right|^2 d\lambda, \tag{7.29}$$

* This problem has been previously considered only for the simplest operators.

** This concept has been introduced by S. G. Mikhlin.

from which, with restriction (7.23), it follows that

$$\int_{-\pi}^{\pi} \left| \frac{\partial \Phi}{\partial \lambda} \right|^2 d\lambda < C.$$

Formula (7.28) enables us to determine the symbol of a singular operator from the characteristic of a singular integral using its Fourier series expansion. It was found possible, however, to sum this series, with the result that an explicit relation was established between these functions:

$$\Phi(q, \lambda) = \frac{1}{2\pi} \int_{\lambda-\pi}^{\pi} \left[\ln \frac{1}{\sin(\lambda-\theta)} + i \frac{\pi}{2} \right] f(q, \theta)\, d\theta + \\ + \frac{1}{2\pi} \int_{\lambda}^{\lambda+\pi} \left[\ln \frac{1}{\sin(\lambda-\theta)} - i \frac{\pi}{2} \right] f(q, \theta)\, d\theta. \tag{7.30}$$

We turn to the simplest operator h. Its symbol is $e^{i\lambda}$. Adjoint to h is the operator $h^* = h^{-1}$. Indeed, to construct an adjoint operator, it is sufficient to transpose the points q_0 and q in the kernel and replace it by its complex conjugate. Consequently, in the case under consideration the symbols of the original and adjoint operators are complex conjugate functions. Representation (7.25) makes it possible to extend this conclusion to the general case.

The discussion up to now has been restricted to the H-L space. Let us show that it is possible to pass to a space of square-summable functions (L_2). We calculate the norms of hu and $h^{-1}u$ in this space.* Consider the integral

$$I = \int_{\Pi} \overline{v(q)}\, v(q)\, dS_q, \quad v(q) = hu = \frac{1}{2\pi} \int_{\Pi} \frac{e^{i\psi_{01}}}{r^2} u(q)\, dS_q. \tag{7.31}$$

We interchange the order of integration and at the same time change the notation on the right-hand side of (7.31):

$$I = \frac{1}{2\pi} \int_{\Pi} u(q_0)\, dS_{q_0} \int_{\Pi} \frac{\overline{v(q_1)}\, e^{i\psi_{01}}}{r^2} dS_{q_1}.$$

Since

$$\frac{1}{2\pi} \int_{\Pi} \overline{v(q_1)} \frac{e^{i\psi_{01}}}{r^2} dS_q = h\bar{v},$$

we have

$$\overline{v(q)} = \frac{1}{2\pi} \int_{\Pi} \overline{u(q_1)} \frac{e^{-i\psi_{01}}}{r^2} dS_{q_1} = -h^1\bar{u}.$$

* u, as before, belongs to the class H-L.

Hence,

$$\frac{1}{2\pi}\int\limits_{\Pi}\overline{v(q_1)}\,\frac{e^{i\psi_{01}}}{r^2}\,dS_{q_1}=-\overline{u(q_0)}.$$

From this we obtain

$$I=\int\limits_{\Pi}|v|^2\,dS_q=\int\limits_{\Pi}|u|^2\,dS_q.$$

We now extend singular operators to the space L_2. Let the function $a(q)$ be square summable. It is then possible to choose a sequence of functions $u_n(q)$ of the class H-L with the convergence in the mean:

$$\lim_{n\to\infty}u_n(q)=u(q).$$

Assume, now,

$$hu=\lim_{n\to\infty}hu_n,\quad h^{-1}u=\lim_{n\to\infty}h^{-1}u_n$$

and similarly for all powers of the operators h and h^{-1}. Thus, the norm of the operator h in the space L_2 is 1. The norm of the operator h^{-1} is calculated in a similar way and is also 1.

We finally define a singular operator (in the space L_2) as a series:

$$Au=\sum_{n=-\infty}^{\infty}a_n(q)\,h^nu.$$

For the following discussion it is very important to establish sufficient conditions on the characteristic of a singular operator under which the operators are bounded in L_2. Suppose that the characteristic satisfies condition (7.23).* We have

$$Au=\sum_{n=-\infty}^{\infty}a_n(q_0)\,h^nu=\sum_{n=-\infty}^{\infty}{}' na_n(q_0)\cdot\frac{1}{n}\,h^nu. \tag{7.32}$$

We now turn to the calculation of the norm $\|Au\|$ using the right-hand side of (7.32):

$$\|Au\|^2=\int\limits_{\Pi}\Big|\sum_{n=-\infty}^{\infty}{}' na_n(q_0)\,\frac{1}{n}\,h^nu\Big|^2 dq_0\leqslant$$

$$\leqslant\int\limits_{\Pi}\Big\{\sum_{n=-\infty}^{\infty}{}' n^2|a_n(q_0)|^2\Big\}\Big\{\sum_{n=-\infty}^{\infty}\frac{1}{n^2}\Big|h^n u\Big|^2\Big\}\,dq_0\leqslant$$

$$\leqslant\sup_{q_0}\sum_{n=-\infty}^{\infty}n^2|a_n(q_0)|^2\int\limits_{\Pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^2}\,|h^nu|^2\,dq_0.$$

* The constructions that follow have been communicated to the authors by S. G. Mikhlin.

Denote the term outside the integral by a constant C_0 and interchange the order of summation and integration:

$$\|Au\|^2 \leqslant C_0 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \int_{\Pi} |h^n u|^2 \, dq_0 = C_0 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \| h^n u \|^2.$$

By making use of the fact that the norms of the operators h and h^{-1} are 1, we obtain

$$\| Au \|^2 < C_0 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \| u \|^2 = C_0 \| u \|^2 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} C_0 \| u \|^2.$$

From the definition of the functions $a_n(q)$ [see (7.24)] it follows that

$$n^2 |a_n(q_0)|^2 = 4\pi^2 |b_n(q_0)|^2.$$

Let us sum these equalities over n and use Parseval's equality:

$$\sum_{n=-\infty}^{\infty} n^2 |a_n(q_0)|^2 = 4\pi^2 \sum_{n=-\infty}^{\infty} |b_n(q_0)|^2 = 4\pi^2 \int_{-\pi}^{\pi} |f(q_0, \theta)|^2 \, d\theta.$$

We arrive at the estimate of the constant C_0 [in terms of the constant appearing in (7.23)]. Thus, it may be considered proved that the operator A is bounded.

Let us also examine the question of the transformation of a symbol when the co-ordinates are changed (I. A. Itskovich [1]). Suppose that the origin of the angle θ is changed at each point q_0 so that $\theta = \theta_1 + \alpha(q_0)$ [where $\alpha(q_0)$ is a continuous function]. The characteristic also changes:

$$f(q_0, \theta) = f(q_0, \theta_1 + \alpha) = f_1(q_0, \theta_1).$$

Let us see how the symbol changes in this case. Suppose that to the characteristic $f(q_0, \theta)$ there corresponds the symbol $\Phi(q_0, \lambda)$, and to the characteristic $f_1(q_0, \theta)$ the symbol $\Phi_1(q_0, \lambda)$. With the new origin of angles and with the change $\theta = \theta_1 + \alpha$, we obtain, according to (7.30), an expression for the symbol $\Phi_1(q_0, \lambda)$ as

$$\Phi_1(q_0, \lambda) = \int_{\lambda+\alpha-\pi}^{\lambda+\alpha} \ln \frac{1}{\sin(\lambda + \alpha - \theta)} [f(q_0, \theta) + f(q_0, \theta + \pi)] \, d\theta +$$

$$+ i \frac{\pi}{2} \int_{\lambda+\alpha-\pi}^{\lambda+\alpha} [f(q_0, \theta) - f(q_0, \theta + \pi)] \, d\theta = \Phi(q_0, \lambda + \alpha).$$

It follows that the range of a symbol is invariant with respect to the way of measuring angles, i.e., unaltered by a change of variables.

The above discussion has been restricted to the case when the surface of integration is a plane Π. We now turn to the case of a closed Lyapunov surface S.

To determine the value of a singular integral, we proceed as follows: assign a number ε and denote by σ_ε a neighbourhood of the point q_0 situated inside a sphere* of radius ε. Since S is a Lyapunov surface, the surface σ_ε can be mapped in a one-to-one manner (for a sufficiently small ε) onto a part of a plane (further denoted by D) so that the transformation at the point q_0 itself is conformal. For example, it is possible to make an orthogonal projection of the region σ_ε on a tangent plane at the point q_0. Henceforth, the projections of the points q_0 and q will be denoted by q_0' and q', respectively.

Based on the above discussion, the transformed integrand is written as

$$K(q_0, q)\,u(q)\,dS_q = K'(q_0', q')\,v(q')\,dS_{q'}, \quad u(q) = v(q'),$$

where the function $K'(q_0', q')$ has the representation

$$K'(q_0', q') = \frac{f(q_0', \theta'')}{r^2} + \frac{f_0(q_0', q')}{r^\mu} \quad (\mu < 2),$$

the function $f_0(q_0', \theta')$ is bounded.

By definition, we assume

$$\int_S u(q)\,K(q_0, q)\,dS_q =$$

$$-\int_{S-\sigma_\varepsilon} u(q)\,K(q_0, q)\,dS_q + \int_D \frac{f(q_0', \theta)}{r^2}\,dS_{q'} + \int_D \frac{f_0(q_0', q')}{r^\mu}\,dS_{q'}. \quad (7.33)$$

The symbol of the singular operator (7.33) will be identified with the symbol corresponding to the characteristic $f(q, \theta)$.** With this definition of a symbol all the foregoing results hold good.

8. Two-dimensional Singular Integral Equations

Consider a singular integral equation

$$Au = a_0(q_0)\,u(q_0) + \int_\Pi K(q_0, q)\,u(q)\,dS_q +$$

$$+ \int_\Pi K_1(q_0, q)\,u(q)\,dS_q = F(q_0) \quad (Au = A'u + Tu = F), \quad (8.1)$$

where the functions $a_0(q)$ and $F(q)$ belong to the class H-L, the kernel $K(q_0, q)$ is of the form $f(q_0, \theta)/r^2$, the kernel $K_1(q_0, q)$ is

* This region can also be determined with the aid of a circular cylinder whose axis coincides with a normal at the point q_0.

** Note that the points θ run over a unit circle situated in a tangent plane.

regular. It is assumed that the characteristic $f(q_0, \theta)$ satisfies condition (7.23).

As in the case of one-dimensional singular equations, the investigation in the two-dimensional case is based on the regularization procedure, i.e., on reducing the original equation to a Fredholm one by applying a specially chosen singular operator (denoted here by $\boldsymbol{B}$) to both its sides. This regularizing operator must be bounded and must have a continuous and bounded symbol.

We impose an important restriction in what follows assuming that the symbol of the operator $\boldsymbol{A}$, i.e., the function $\Phi(q_0, \lambda)$ does not vanish for any combinations of q_0 and λ, including $q_0 = \lambda$. The function

$$\Phi'(q_0, \lambda) = \frac{1}{\Phi(q_0, \lambda)} \tag{8.2}$$

is then bounded and continuous and, as can easily be verified, has a derivative satisfying the condition

$$\int \left| \frac{\partial \Phi}{\partial \lambda} \right|^2 d\lambda < C.$$

Let $\Phi'(q_0, \lambda)$ be considered as the symbol of some singular operator. The explicit expression for its characteristic can be obtained by using the expansion of the symbol $\Phi'(q_0, \lambda)$ in a Fourier series and subsequently determining the coefficients of the characteristic according to (7.24). It can be shown that the characteristic satisfies condition (7.23) [of course, with a different value of the constant than for the characteristic $f(q_0, \lambda)$]. The operator thus introduced is a regularizing one. Indeed, according to the properties of the symbol of a composition proved above, the symbol of the operator

$$\boldsymbol{BA} = \boldsymbol{BA}' + \boldsymbol{BT}$$

is 1 (the operator $\boldsymbol{BT}$ is completely continuous)a,nd hence the equation

$$\boldsymbol{BA}u = \boldsymbol{B}F$$

is a Fredholm one.

As before (see Sec. 4), the index is defined as the difference in the number of linearly independent solutions of the original and companion equations. By using the general considerations of the concluding part of Sec. 4, it can be shown that the equation $\boldsymbol{A}u = F$ is solvable when its right-hand side is orthogonal to all solutions of the companion equation.

The regularization thus performed is not, in general, equivalent. In a number of cases, however, it can be shown that the additional regular term for the regularizator can be chosen so as to ensure equivalency. We begin the proof with the consideration of the simplest

equation

$$a^1(q_0)\,u(q_0) - \int\limits_{\Pi} \frac{e^{i\psi}}{r^2}\,u(q)\,dS_q = F(q_0), \qquad [a^1(q) - h]\,u = F(q_0). \tag{8.3}$$

The condition that the symbol is not zero means, in reference to Eq. (8.3), the fulfilment of the inequality $|a^1(q)| \neq 1$. In view of the continuity of the coefficient $a^1(q)$, the inequality $|a^1(q)| < 1$ or $|a^1(q)| > 1$ must be fulfilled. Assume that $|a^1(q)| > 1$. Equation (8.3) is then represented as

$$u(q) = \frac{1}{a^1(q)}\,hu + \frac{F(q)}{a^1(q)}. \tag{8.4}$$

This equation can be solved by the method of successive approximations, which leads [using identity (7.26)] to a representation of the solution in the form

$$u(q) = \sum_{n=0}^{\infty} \left[\frac{h}{a^1(q)}\right]^n \frac{F(q)}{a^1(q)} = [a^1(q) - h]^{-1} F(q) + T_1^* F, \tag{8.5}$$

where

$$T_1^* F = \sum_{n=2}^{\infty} \left\{\left[\frac{h}{a^1(q)}\right]^n - \frac{h^n}{[a^1(q)]^n}\right\} \frac{F(q)}{a^1(q)}.$$

If the right-hand side of Eq. (8.4) includes also the regular operator (previously transposed from the left-hand side), expression (8.5) is a Fredholm equation equivalent to the original one.

Assume now that $|a^1(q)| < 1$. By applying the operator h^{-1} to both sides of Eq. (8.3), we arrive at an equivalent equation, namely

$$u(q) = a^1(q)\,h^{-1}u - h^{-1}F. \tag{8.6}$$

Equation (8.6) can be solved by the method of successive approximations, which leads, in the presence of an additional regular operator, to equivalent regularization.

We now proceed to the consideration of a more general case when the symbol of a singular operator is a trigonometric polynomial and the operator itself can therefore be represented as a product:

$$a_0^1(q)\,h^m \prod_{k-1}^{n} (a_k^1 - h)\,u = F(q), \tag{8.7}$$

where m, n are integers, with $n > 0$, a_k^1 are some functions.

From the condition that the symbol is not zero we have the inequalities $|a_k^1(q)| \neq 1$ $(k = 1, 2, \ldots, n)$. It is readily seen that Eq. (8.7) is reduced to an equivalent Fredholm equation by an oper-

ator that is symbolically written as

$$\frac{1}{a_0^1(q)} h^{-m} \prod_{k=1}^{n} \{[a_k^1(q)-h]^{-1}+T_k^*\}, \tag{8.8}$$

where T_k^* are previously defined completely continuous operators.

We proceed to the general case. The operator A is represented as a series, namely

$$A = \sum_{k=-\infty}^{\infty} a_k(q) h^k. \tag{8.9}$$

Since the symbol of the operator A is assumed to be different from zero, it follows that there is a large number n such that the operator

$$A_n = \sum_{k=-n}^{n} a_k(q) h^k$$

has a non-zero symbol, and hence there is a regularizing operator B_n corresponding to it.

It is obvious that the following limiting equalities are fulfilled:

$$\lim_{n\to\infty} \| B_n \| = \| B \|, \quad \lim_{n\to\infty} \| r_n \| = 0,$$

where r_n is the sum of the remaining terms in (8.9).

We now represent Eq. (8.1) as

$$A_n u = r_n u + F(q).$$

By factorizing the symbol of the operator A_n, we determine the regularizing operator B_n (with a suitable selection of the operators T_k^*). We arrive at the equation

$$u(q) = B_n(q)\, r_n(q)\, u + F_1(q), \tag{8.10}$$

which is solvable by the method of successive approximations by virtue of a special choice of the number n [the meaning of $F_1(q)$ is obvious].

Thus,it is proved that under the specified conditions on the characteristic the singular integral equation admits an equivalent regularization, and hence its index is always zero. Note that the result proved above is not extended automatically to the case of systems of two-dimensional singular equations (for details, see below).

The foregoing proof that the index of a singular integral equation is zero has been carried out for a plane Π. Its extension to the case of an arbitrary Lyapunov surface is difficult to achieve since it is not always possible to choose a regular co-ordinate network on the surface and to obtain a unique representation of the symbol suitable on the whole surface.

Below is a different proof of the possibility of an equivalent regularization free of this shortcoming. We introduce a complex parameter into the singular equation (8.1):

$$u(q_0) + \nu \int_{\Pi} K(q_0, q)\, u(q)\, dS_q + Tu = F(q_0). \qquad (8.11)$$

In this equation the normalization has been performed with respect to the coefficient $a_0(q_0)$. To the original equation corresponds the number $\nu = 1$, and to the Fredholm equation $\nu = 0$.

Let us now establish regions in the ν plane in which the minimum value of $|1 - \nu\Psi(q, \lambda)|$ is zero* (i.e., for each ν from this region there is a pair of values q_0 and λ for which this minimum is zero). The complementary part of the plane is the sum of a finite or countable number of regions which are denoted by δ_j. Let us prove that in any one of the regions δ_j the index of Eq. (8.11) remains constant. In each of the regions the modulus of the symbol is different from zero (i.e., $|1 - \nu\Phi(q, \lambda)| > 0$), and hence there exists a regularizing operator, which is denoted by B_ν.** The symbol of this regularizator is $[1 - \nu\Phi(q, \lambda)]^{-1}$.

Consider also an equation with a parameter $\nu + \Delta\nu$ (where $\Delta\nu$ is a sufficiently small quantity chosen in such a way that $\nu + \Delta\nu$ belongs to the same region δ_j). The equation is re-written as

$$u - \nu K u = Tu + \Delta\nu\, Ku + F.$$

By applying the regularizator B_ν to both sides, we arrive at the equation

$$u - \Delta\nu\, Lu = B_\nu F + T_1 u,$$

where T_1 is a completely continuous operator. The singular operator L has the symbol $(1 - \lambda\Phi)^{-1}$ which is bounded. The norm of the operator $\Delta\nu L$ can therefore be made arbitrarily small by a suitable choice of $\Delta\nu$. The last equation can then be reduced to an equivalent Fredholm equation by the method of successive approximations. Consequently, the indices of the operators $I - \nu K$ and $I - (\nu + \Delta\nu) K$ must be identical.

The foregoing allows us to ascertain at once that the index is zero when the function $\Psi(q, \lambda)$ is strictly less than 1 in modulus. It is obvious that one of the regions δ_j includes a circle $|\nu| < 1 + \varepsilon$. The equation therefore has the same index (equal to zero) when $\nu = 1$ and $\nu = 0$.

Consider, now, the question of equivalent regularization by a procedure suitable also for the case of closed surfaces. The original singular equation is written as (on introducing a complex parameter

* In this case the function $\Psi(q, \lambda)$ is the symbol of the singular term only.

** Here we do not mean an equivalent regularization.

ν into it in a special way)

$$u - \nu (u - Au) + Tu = F. \tag{8.12}$$

In addition, we require that the symbol of the operator should be continuous in q on the surface S uniformly with respect to θ and that it should have the third square-summable derivatives. Assume that the function $F(q)$ belongs to the class L_2.

Suppose that there is a curve L in the ν_1 plane with ends at zero and infinity, having no common points with the range of the symbol. There must therefore be a constant β such that for all points q and all λ we have the inequality

$$|\Phi(q, \lambda) - \nu_1| > \beta, \quad \nu_1 \in L. \tag{8.13}$$

Denote by $\widetilde{L}$ a curve into which the curve L is mapped by the transformation $\nu_1 = (\nu - 1)/\nu$. It is obvious that the curve $\widetilde{L}$ can be chosen so that it will not pass through the point 1, and hence the curve will be bounded. Its ends are the points 0 and 1.

Consider, now, the singular equation (8.12) assuming that the point ν is situated on the line $\widetilde{L}$. Let us prove that the symbol of Eq. (8.12) (the function $1 - \nu + \nu\Phi$) is bounded from below in modulus. Indeed, by (8.13),

$$\left|\frac{1-\nu+\nu\Phi}{\nu}\right| = \left|\Phi(q, \lambda) - \frac{\nu-1}{\nu}\right| \geqslant \beta.$$

Since the symbol $\Phi(q, \lambda)$ is bounded, there is a constant $K > |1 - \Phi|$. For the values $|\lambda| \geqslant 1/2K$ we have the inequality

$$|1-\nu+\nu\Phi| \geqslant \frac{\beta}{2K}.$$

If $|\lambda| < 1/2K$, we obtain

$$|1 - \nu + \nu\Phi| > 1 - |\nu| K > 1/2.$$

By using the previously established property of the index to be constant in the regions δ_j, we arrive at the proof that the index of the original equation, i.e. Eq. (8.12) with $\nu = 1$, is zero (when $\nu = 0$, the equation is a Fredholm one).

The proposed method reduces the question of equivalent regularization to the proof of the existence of the line L $(\widetilde{L})$ introduced above. Since the discussion is carried out in the space L_2, we have to pass to an equation

$$(A^* + T^*)(A + T)u = (A^* + T^*)F,$$

equivalent to the original one (see Sec. 4). We obtain an equation whose symbol is $|\Phi(q, \lambda)|^2$, and from the condition that the function $\Phi(q, \lambda)$ is not zero it follows that the line L can be the negative semiaxis.

We describe a scheme for the explicit construction of an operator to effect an equivalent regularization using the boundedness of the line $\tilde{L}$. It has been proved above that $|1 - \nu + \nu\Phi| > 0$. Hence an operator whose symbol is $\frac{1 - \Phi(q, \lambda)}{1 - \nu + \nu\Phi(q, \lambda)}$ is bounded by a certain constant C independent of ν, and consequently $C \geqslant \| I - A \|$. Assign $|\nu_0| = 1/2C$. The operator $[I - \nu_0 (I - A)]^{-1}$ is then bounded. By applying this operator to both sides of Eq. (8.12), we obtain an equivalent equation with the symbol

$$\frac{1-\nu+\nu\Phi}{1-\nu_0+\nu_0\Phi} = 1 - \frac{(\nu-\nu_0)(1-\Phi)}{1-\nu_0+\nu_0\Phi}.$$

We operate on both sides of the new equation with an operator whose symbol is $(1 - \nu_0 + \nu_0\Phi)(1 - \nu_1 + \nu_1\Phi)^{-1}$, with $|\nu_1 - \nu_0| \leqslant \leqslant 1/2C$. The result is an equivalent equation with the symbol

$$1 - \frac{(\nu-\nu_1)(1-\Phi)}{1-\nu_1+\nu_1\Phi}.$$

By repeating this process, after the kth step we arrive at an equation with the symbol

$$1 - \frac{(\nu-\nu_k)(1-\Phi)}{1-\nu_k+\nu_k\Phi}.$$

By choosing a sufficiently large k, we can obtain $\nu_k = 1$, as required.

The foregoing theory can be extended to the case of systems of singular equations. Consider systems of the form

$$\sum_{k=1}^{n} A_{jk}u_k = F_j \quad (j = 1, 2, \ldots, n), \tag{8.14}$$

$$A_{jk}u_k = a_{jk}(q_0)\, u_k(q_0) + \int_S u_k(q) \frac{f_{jk}(q_0, \theta)}{r^2}\, qS_q + T_{jk}u_k,$$

where T_{jk} are completely continuous operators.

The concept of the symbol of a singular operator introduced above is extended to the case of systems leading to a so-called *symbolic matrix* $\| \Phi_{jk} \|$. The determinant of this matrix is called the symbolic determinant of system (8.14). It is assumed that all elements of a symbolic matrix satisfy the same conditions as for the symbol of a single equation.

It can be shown that the composition of singular operators is performed by the rules of matrix algebra, with the multiplication of symbolic matrices.

We require that a symbolic determinant should be different from zero. We form a matrix with elements $\Phi'_{jk} = \Phi^{jk}\Phi^{-1}$ (where Φ^{jk} is the cofactor of the element Φ_{jk}). It can be shown that a matrix

operator whose symbolic matrix is composed of the elements Φ'_{jk} is a regularizing operator for the original system (8.14).

We now turn to the consideration of the question of equivalent regularization. Assume that the symbolic matrix is of the form

$$\Phi(q, \lambda) = E - \Psi(q, \lambda).$$

Suppose that the characteristic numbers of the matrix $\Psi(q, \lambda)$ are less than 1 in modulus for all q and λ. Let us determine regions δ_j in the ν plane such that the characteristic numbers of the matrix $E - \nu\Psi(q, \lambda)$ are different from zero at their points. By the argument used in the case of a single equation it can be shown that these regions are also the regions of constancy of the index. From the restriction imposed on the arrangement of the characteristic numbers of the matrix $\Psi(q, \lambda)$ it follows that the circle $|\nu| < 1 + \varepsilon$ is situated in one of these regions. Consequently, the index of the original system ($\nu = 1$) is zero.

In contrast to the case of a single equation, the condition of non-vanishing symbolic matrix is not, in general, sufficient for equivalent regularization. It is possible, however, to determine certain classes of systems of equations for which an equivalent regularization can be effected.It is most essential to the questions investigated in the monograph that these classes include singular integral equations of the theory of elasticity (see Sec. 29). We therefore restrict our study to these cases.

Assume that there is a curve L in the ν_1 plane joining the points 0 and ∞ and having no common points with the set of characteristic numbers of the matrix $\Phi(q, \lambda)$. As in the case of a single equation, we introduce the parameter $\nu = 1/(1 - \nu_1)$.

Consider, now, the equation

$$u - \nu(I - A)u = F, \tag{8.15}$$

where I is the identity operator. The symbolic matrix of this operator is of the form

$$E - \nu(E - \Phi) = \nu(\Phi - \nu_1 E).$$

Let us prove that the determinant of this matrix is different from zero for $\nu_1 \in L$ (or $\nu \in L$). If $|\nu|$ is small, the determinant is close to $|1 - \nu|^n$ (n is the order of the system), and for a given $\xi > 0$ it is possible to choose a value $\eta > 0$ such that the following inequality is fulfilled:

$$|||(1 - \nu)E + \nu\Phi||| > \xi \text{ if } |\nu| < \eta.$$

Let $|\nu| > \eta$. We then have

$$|\,||(1 - \nu)E - \nu\Phi||\,| > \eta^n |\,||\nu_1 E - \Phi||\,| = \eta^n \prod_{h=1}^{n} \left| \nu_1 - \nu_1^k(q, \lambda) \right|.$$

where ν_1^k are the characteristic numbers of the matrix Φ. Since there exists a curve L, it follows that there is a constant γ such that $|\nu_1 - \nu_1^k(q, \lambda)| \geqslant \gamma$ $(k = 1, 2, \ldots, n)$. We finally obtain

$$|||(1-\nu)E+\nu\Phi||| \geqslant (\gamma\eta)^n \quad (\nu \in \widetilde{L},\ |\nu| \geqslant \eta).$$

It can therefore be shown that in the space L_2 there exists a bounded operator whose symbolic matrix is of the form

$$[(1 - \nu)E + \nu\Phi(q, \lambda)]^{-1}[E - \Phi(q, \lambda)].$$

Let this operator be denoted by H_ν and suppose that its norm does not exceed a constant C. Then $C > \| I - A \|$. For $\nu_0 \in \widetilde{L}$ and $|\nu_0| \leqslant 1/2C$ the operator $[I - \nu_0(I - A)]^{-1}$ is therefore bounded and has a zero index. By applying this operator to both sides of Eq. (8.15), we arrive at an equivalent equation having the symbolic matrix

$$[(1-\nu_0)E+\nu_0\Phi]^{-1}[(1-\nu)E+\nu\Phi] =$$
$$= E-(\nu-\nu_0)[(1-\nu_0)E+\nu_0\Phi]^{-1}(E-\Phi).$$

By operating on both sides of this equation with the operator

$$[I-(\nu_1-\nu_0)H\nu_0]^{-1} \quad \left(\nu_1 \in \widetilde{L},\ |\nu_1-\nu_0| \leqslant \frac{1}{2C}\right),$$

we again arrive at an equivalent equation having, now, the symbolic matrix

$$E-(\nu-\nu_1)[(1-\nu_1)E+\nu_1\Phi]^{-1}(1-\Phi).$$

We repeat the proposed process for a set of values ν_k such that $|\nu_k - \nu_{k-1}| < 1/2C$. By choosing n sufficiently large, we arrive at the value $\nu_k = 1$ corresponding to the original equation $Au = F$.

Consequently, the Fredholm alternatives are applicable to this system of singular equations if the symbolic determinant is different from zero and there is a curve L $(\widetilde{L})$ having the properties specified above.

The existence of a line L is quite obvious when the symbolic matrix is a Hermitian (i.e. self-conjugate) matrix since its characteristic numbers are real, and so L can be taken coinciding with the imaginary semiaxis.

Chapter II

APPROXIMATE METHODS FOR SOLVING INTEGRAL EQUATIONS

9. General Principles of the Theory of Approximate Methods

The approximate methods for solving integral equations are covered adequately by the general theory of approximate methods, and a more detailed discussion of any particular method is only needed to obtain certain estimates. We therefore present the relevant general theory following the work of L. V. Kantorovich [2].

Let x and y be the elements of normed spaces X and Y, and let $\boldsymbol{K}$ be a linear operator mapping X into Y. Consider the functional equation

$$\boldsymbol{K}x = (\boldsymbol{E} - \lambda \boldsymbol{H})\, x = y. \tag{9.1}$$

In addition to Eq. (9.1), we also consider an "approximate equation"

$$\overline{\boldsymbol{K}}\overline{x} = (\overline{\boldsymbol{E}} - \lambda\overline{\boldsymbol{H}})\overline{x} = \overline{y}, \tag{9.2}$$

where $\overline{x}$, $\overline{y}$ are the elements of the spaces $\overline{X}$ and $\overline{Y}$, the operator $\overline{\boldsymbol{K}}$ maps $\overline{X}$ into $\overline{Y}$, $\boldsymbol{E}$ and $\overline{\boldsymbol{E}}$ are respectively identity operators. The spaces $\overline{X}$ and $\overline{Y}$ are chosen simpler in a certain sense than X and Y, and the operator $\overline{\boldsymbol{K}}$ is assumed to be close (also in a certain sense) to the operator $\boldsymbol{K}$.

Subsequently two theorems will be proved, providing a basis for judging the solvability of Eq. (9.2) if Eq. (9.1) is solvable, and the degree of closeness of the solutions of these equations. And for the present let us consider linear normed spaces X and $\overline{X}$ assuming $\overline{X}$ to be complete and isomorphic to some subspace $X' \subset X$. Suppose that the isomorphism is effected by an operator φ_0 (X' into $\overline{X}$) having a continuous inverse operator φ_0^{-1}. Suppose also that there is an operator φ mapping X into $\overline{X}$ and coinciding with φ_0 on X'.

The requirement of "closeness" of Eqs. (9.1) and (9.2) is expressed by the condition

$$\| \varphi \boldsymbol{H} x' - \overline{\boldsymbol{H}} \varphi x' \| \leqslant \varepsilon \| x' \| \quad (x' \in X'), \tag{9.3}$$

or in an alternate form:

$$\| \varphi K x' - \bar{K}\varphi x' \| < \varepsilon \mid \lambda \mid \cdot \| x' \|, \tag{9.3'}$$

where ε is an arbitrarily small positive number.

Let the operator K have an inverse operator, and let the solution of Eq. (9.1) be $x^* \in X'$. Let the operator $\bar{K}$ also have an inverse operator, and let the solution of Eq. (9.2) be $\bar{x}_0$. We prove that the following estimate holds:

$$\| \bar{x}_0 - \varphi x^* \| < \varepsilon |\lambda| \cdot \| \bar{K}^{-1} \| \cdot \| x^* \|. \tag{9.4}$$

Indeed,

$$\| \bar{K}\varphi x^* - \bar{K}\bar{x}_0 \| = \| \bar{K}\varphi x^* - \varphi K x^* \| < \varepsilon |\lambda| \cdot \| x^* \|,$$

since $\bar{K}\bar{x}_0 = \bar{y} = \varphi y = \varphi K x^*$, from which we obtain

$$\| \varphi x^* = \bar{x}_0 \| = \| \bar{K}^{-1}\bar{K}(\varphi x^* - \bar{x}_0) \| < \varepsilon |\lambda| \cdot \| x^* \| \cdot \| \bar{K}^{-1} \|.$$

We require in addition that for every element $x \in X$ there should be an element $x' \in X'$ such that

$$\| Hx - x' \| \leqslant \varepsilon_1 \| x \|. \tag{9.5}$$

We now proceed to prove the theorems.

(I) If estimates (9.3), (9.5) hold and if, further, the operator K^{-1} exists, and on condition that the constant

$$q = [\, \| \varphi \| \cdot \| K \| \cdot |\lambda| \varepsilon_1 + |\lambda| \varepsilon + |\lambda^2| \varepsilon\varepsilon_1] \cdot \| K^{-1} \| \cdot \| \varphi_0^{-1} \| < 1, \tag{9.6}$$

Eq. (9.2) always has a solution, and

$$\| \bar{x} \| \leqslant \frac{(1 + |\lambda| \varepsilon_1) \| K^{-1} \| \cdot \| \varphi \| \cdot \| \varphi_0^{-1} \|}{1 - q} \| \bar{y} \|. \tag{9.7}$$

Let the element $\bar{y}$ be denoted by $\bar{y}_0$, and the element $\varphi_0^{-1}\bar{y}_0$ by y_0. This indexing is convenient in constructing successive approximations. We introduce the notation

$$z = K^{-1}\lambda H y_0. \tag{9.8}$$

It is obvious that

$$Kz = z - \lambda Hz = \lambda H y_0, \quad z = \lambda H (z + y_0). \tag{9.9}$$

Let us find, according to (9.5), an element $x' \in X'$ to fulfil the inequality

$$\| H(z + y_0) - \frac{x'}{\lambda} \| < \varepsilon_1 \| z + y_0 \|.$$

By (9.9),

$$\| z - x' \| < \varepsilon_1 \mid \lambda \mid \cdot \| z + y_0 \|. \tag{9.10}$$

Since

$$K(z + y_0) = z + y_0 - \lambda H (z + y_0) = y_0,$$

it follows that

$$\| z+y_0 \| \leqslant \| K^{-1} \| \cdot \| y_0 \|. \tag{9.11}$$

We finally obtain

$$\| z-x' \| \leqslant \varepsilon_1 |\lambda| \| K^{-1} \| \cdot \| y_0 \|. \tag{9.12}$$

We introduce an element

$$\bar{x}_1 = \varphi (x' + y_0)$$

and estimate the difference:

$$\| \bar{K}\bar{x}_1 - y_0 \| \leqslant \| \bar{K}\varphi (x'+y_0) - \varphi K (x'+y_0) \| + \| \varphi K (x'+y_0) \| +$$
$$+ \| \varphi K (x'+y_0) - \varphi y_0 \| = \| \bar{K}\varphi (x'+y_0) - \varphi K (x'+y_0) \| +$$
$$+ \| \varphi\lambda H y_0 - \varphi K x' \|.$$

From (9.3′) and (9.8) we obtain

$$\| \bar{K}\bar{x}_1 - \bar{y}_0 \| \leqslant \varepsilon |\lambda| \cdot \| x'+y_0 \| + \| \varphi K z - \varphi K x' \|.$$

Taking into account (9.10) and (9.11), we have

$$\| x'+y_0 \| \leqslant \| z+y_0 \| + \| z-x' \| \leqslant (1+\varepsilon_1 |\lambda|) \| K^{-1} \| \cdot \| y_0 \| \leqslant$$
$$\leqslant (1+\varepsilon_1 |\lambda|) \| K^{-1} \| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{y}_0 \|. \tag{9.13}$$

By using (9.12), we arrive at the estimate

$$\| \bar{K}\bar{x}_1 - \bar{y}_0 \| < \varepsilon |\lambda| (1+\varepsilon_1 |\lambda|) \| K^{-1} \| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{y}_0 \| +$$
$$+ \| \varphi \| \cdot \| K \| \varepsilon_1 |\lambda| \cdot \| K^{-1} \| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{y}_0 \| = \{ \| \varphi \| \cdot \| K \| \cdot |\lambda| \varepsilon_1 +$$
$$+ |\lambda| \varepsilon + |\lambda^2| \varepsilon\varepsilon_1 \} \| K^{-1} \| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{y}_0 \| = q \| \bar{y}_0 \|. \tag{9.14}$$

Thus, the element $\bar{x}_1$ satisfies Eq. (9.2) to within $q \| \bar{y}_0 \|$ (to better than the identical zero in which case the right-hand side is equal to $\bar{y}_0$).

We now implement the method of successive approximations. Assume $\bar{y}_1 = \bar{y}_0 - \bar{K}\bar{x}_1$. By (9.14),

$$\bar{K}\bar{x}_1 = \bar{y}_0 - \bar{y}_1, \quad \| \bar{y}_1 \| < q \| \bar{y}_0 \|,$$

and because of (9.13)

$$\| \bar{x}_1 \| = \| \varphi (x'+y_0) \| \leqslant \{ \| \varphi \| (1+\varepsilon_1 |\lambda|) \| K^{-1} \| \cdot \| \varphi_0^{-1} \| \} \| \bar{y}_0 \| =$$
$$= M \| \bar{y}_0 \|.$$

Just as the element $\bar{x}_1$ is constructed from y_0, so we find $\bar{x}_2$ proceeding from y_1 and determine $\bar{y}_2 = \bar{y}_1 - \bar{K}\bar{x}_2$. We have

$$\| x_2 \| \leqslant M \| y_1 \| \leqslant Mq \| \bar{y}_0 \|, \quad \| \bar{y}_2 \| \leqslant q \| \bar{y}_1 \| \leqslant q^2 \| y_0 \|.$$

In a similar way we determine successively $\bar{x}_n$ and $\bar{y}_n$:

$$\bar{K}\bar{x}_n = \bar{y}_{n-1} - \bar{y}_n. \tag{9.15}$$

The following estimates hold:

$$\| \bar{x}_n \| < Mq^{n-1} \| \bar{y}_0 \|, \quad \| \bar{y}_0 \| \leqslant q^n \| \bar{y}_0 \|.$$

By adding equalities (9.15) together, we find

$$\bar{K} \left(\sum_{n=1}^{\infty} \bar{x}_n \right) = \bar{y}_0 - \bar{y}_n,$$

the series $\sum_{n=1}^{\infty} \bar{x}_n$ converges since $q < 1$. Denote its sum by x. The element $x \in \bar{X}$ since the space $\bar{X}$ is complete, and the element $\bar{x}$ satisfies the equation

$$\bar{K}\bar{x} = \bar{y}_0 = \bar{y} \quad \left(\| \bar{x} \| < \frac{M}{1-q} \| \bar{y} \| \right),$$

which was to be proved.

We now introduce a further condition that the right-hand side of Eq. (9.1), viz. the element y, admits approximation by the element $y' \in X'$ in such a way that

$$\| y - y' \| < \varepsilon_2 \| y \|. \tag{9.16}$$

Determine, according to (9.5), the element $x' \in X'$ so that

$$\| Hx^* - x' \| \leqslant \varepsilon_1 \| x^* \|.$$

Then

$$\| x^* - (\lambda x' + y') \| \leqslant$$
$$\leqslant \| y - y' \| + |\lambda| \cdot \| x^* - \lambda x' \| < \varepsilon_2 \| y \| + |\lambda| \varepsilon_1 \| x^* \| \leqslant$$
$$\leqslant |\lambda| (\varepsilon_1 + \varepsilon_2 \| K \|) \| x^* \| = \eta \| x^* \|. \tag{9.17}$$

Let us prove the second theorem concerning the solvability of Eq. (9.2) if Eq. (9.1) is solvable.

(II) If conditions (9.3), (9.5), and (9.16) are fulfilled and if there exists an operator $\bar{K}^{-1}$, the following estimate holds:

$$\| x^* - \varphi_0^{-1} \bar{x}_0 \| \leqslant \{ 2\varepsilon |\lambda| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{K}^{-1} \| +$$
$$+ (\varepsilon_1 |\lambda| + \varepsilon_2 \| K \|)(1 + \| \varphi K \| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{K}^{-1} \|) \} \| x^* \| = p \| x \|^*, \tag{9.18}$$

where $\bar{x}_0$ is the solution of Eq. (9.2).

According to (9.17), there is an element $x' \in X'$ such that

$$\| x^* - x' \| \leqslant \eta \| x^* \|. \tag{9.19}$$

Denote by $\bar{x}_1$ the solution of the equation $\overline{Kx} = \varphi K x'$. Since $\bar{x}_0$ is the solution of Eq. (9.2), it follows that

$$\bar{K}\bar{x}_0 = \varphi K x^*, \quad \bar{K}(\bar{x}_0 - \bar{x}_1) = \varphi K (x^* - x');$$

from this we obtain

$$\| \bar{x}_0 - \bar{x}_1 \| \leqslant \| \bar{K}^{-1} \| \cdot \| \varphi K \| \eta \| x^* \|. \tag{9.20}$$

Since $x' \in X'$, condition (9.4) is fulfilled:

$$\| \bar{x}_1 - \varphi x' \| < \varepsilon |\lambda| \, \| \bar{K}^{-1} \| \cdot \| x' \| \leqslant \varepsilon |\lambda| (1+\eta) \| \bar{K}^{-1} \| \cdot \| x^* \|,$$

and hence

$$\| \varphi_0^{-1} \bar{x}_1 - x' \| \leqslant \varepsilon |\lambda| (1+\eta) \| \varphi_0^{-1} \| \cdot \| \bar{K}^{-1} \| \cdot \| x^* \|. \tag{9.21}$$

It follows from (9.20) that

$$\| \varphi_0^{-1} \bar{x}_1 - \varphi_0^{-1} \bar{x}_0 \| \leqslant \| \varphi_0^{-1} \| \cdot \| \bar{K}^{-1} \| \cdot \| \varphi K \| \cdot \| x^* \|. \tag{9.22}$$

On comparing (9.19), (9.21), and (9.22), we obtain (the inequality is strengthened by replacing η in the first term by unity)

$$\| x^* - \varphi_0^{-1} \bar{x}_0 \| < \{ 2\varepsilon |\lambda| \cdot \| \varphi_0^{-1} \| \cdot \| \bar{K}^{-1} \| + \eta (1 + \| \varphi_0^{-1} \| \times$$
$$\times \| \bar{K}_0^{-1} \| \cdot \| \varphi K \|) \} \| x^* \|.$$

Estimate (9.18) follows from this, which was to be proved.

The procedure of constructing an approximate solution of the operator equation (9.1) consists in choosing a family of approximate operator equations and solving each of them. The convergence of the solutions of these equations to the exact one is achieved because of the fact that the coefficient p [see (9.18)] can be made arbitrarily small.

Let us now discuss one more question of the theory of approximate methods, the interpolation of functions. Suppose that a solution is obtained in the space X' in the course of the solution of an approximate operator equation of the second kind. It is required, however, to determine an (approximate) solution in the space X.

Different approaches are possible. We may use, for example, the representation

$$x = y + \lambda H \bar{x}'. \tag{9.23}$$

But we may use any interpolation procedures such as a piecewise linear interpolation, Lagrange's polynomials. For definiteness, we assume that a network x_i ($i = 0, \ldots, n$, $x_0 = a$, $x_n = b$) is given on a segment of the real axis and that values of a function $f(x)$ whose differential properties have been established are known at the nodes of this network. A disadvantage of Lagrange's polynomials, for example, is that a considerable oscillation of the representation is possible for a sufficiently large number n.

In recent years extensive use has been made of new mathematical technique involving so-called splines (N. S. Bakhvalov [1], G. I. Marchuk [1]), which does not have this disadvantage. Suppose that it is required to find, on the same segment $[a, b]$, a function $g(x)$ belonging to the class C^k (i.e., continuous together with its derivatives up to the order k). This function on each of the segments $[x_{i-1}, x_i]$ is a polynomial of degree $k+1$:

$$g(x) \equiv g_i(x) = \sum_{l=0}^{k+1} a_l^i (x_i - x)^l \quad (i = 1, 2, \ldots, n). \tag{9.24}$$

At the points x_i of the network we have the equalities

$$g(x_i) = f_i \tag{9.25}$$

and the following boundary conditions:

$$g^{(k)}(a) = g^{(k)}(b) = 0. \tag{9.26}$$

Since the function $g(x)$ is continuous and has continuous derivatives of order k, there are also additional equalities at the points x_i. For simplicity, we restrict our study to the case of cubic splines. The specified conditions then become

$$\begin{aligned} &g_{i+1}(x_i) = g_i(x_i), \quad g'_{i+1}(x_i) = g'_i(x_i), \\ &g''_{i+1}(x_i) = g''_i(x_i) \quad (i = 1, 2, \ldots, n-1). \end{aligned} \tag{9.27}$$

Thus, to determine the unknowns a_l^i we have a system of equations, the corresponding matrix being a three-diagonal one:

$$\left\| \begin{matrix} 2(h_1+h_2) & h_2 & 0 & 0 & 0 & \ldots & & \\ h_2 & 2(h_2+h_3) & h_3 & 0 & 0 & \ldots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & \ldots & & h_{n-1} & 2(h_{n-1})+h_n \end{matrix} \right\|, \tag{9.28}$$

where $h_i = x_{i+1} - x_i$.

For a numerical solution of this system it is convenient to apply the orthogonal polynomial method which consists in successively calculating auxiliary quantities:

$$\begin{aligned} \beta_i &= \begin{cases} -\dfrac{h_2}{2(h+h_2)} & (i=1), \\ \dfrac{-h_{i+1}}{2(h_i+h_{i+1})+h_i\beta_{i-1}} & (i>1), \end{cases} \\ Z_i &= \begin{cases} \dfrac{F_1}{2(h_1+h_2)} & (i=1), \\ \dfrac{F_i - h_i Z_{i-1}}{2(h_i+h_{i+1})+h_i\beta_{i-1}} & (i>1), \end{cases} \\ a_2^{(i)} &= \begin{cases} Z_{n-1} & (i=n-1) \\ \beta_i a_2^{(i+1)} + Z_i & (1 \leqslant i < n-1), \end{cases} \end{aligned} \tag{9.29}$$

where F_i are the right-hand sides of the system.

After calculating $a_2^{(i)}$ $(i < n-1)$ the coefficients $a_1^{(i)}$ and $a_3^{(i)}$ are determined from the formulas

$$\left.\begin{aligned} a_1^i &= -\frac{h_3}{3}(a_2^{i-1}+2a_2^i) - \frac{f_{i-1}-f_i}{h_i} \\ a_3^i &= \frac{a_2^{i-1}-a_2^i}{3h_i}. \end{aligned}\right\} \quad (i=1,\ 2,\ \ldots,\ n), \qquad (9.30)$$

The coefficients $A_0^{(i)}$ are immediately found from equalities (9.25).

Let us show that the spline approximation has definite advantages. Suppose that there is a piecewise linear approximation. This function is the solution of a variational problem (with restrictions)

$$\min\ \Phi(u) = \int_a^b \left(\frac{du}{dx}\right)^2 dx, u(x_i) = f_i \quad (i=0,\ 1,\ \ldots,\ n) \qquad (9.31)$$

in the class of functions having square-summable first derivatives.

The approximation by cubic splines solves the problem of minimizing the functional

$$\Phi(u) = \int_a^b \left(\frac{d^2u}{dx^2}\right)^2 dx, \qquad u(x_i) = f_i \quad (i=0,\ 1,\ \ldots,\ n) \qquad (9.32)$$

in the corresponding class.

Note, also, that the error of the interpolation by cubic splines has the estimate

$$\max |g(x) - f(x)| \leqslant Ch^{\alpha},$$

where $h = \max h_i$ and α and C are constants. If $f(x)$ has continuous second derivatives, then $\alpha = 2$.

The approximation for the unknown function by splines in reference to the solution of integral equations has been considered in a number of works (see, for example, J. H. Ahlberg, E. N. Nilson, J. L. Walsh [1]).

With regard to the advisability of the approximation for the solution of an integral equation already obtained (in some way) the following may be said. The necessity for this kind of additional procedure for particular physical problems is dictated by the fact that it is not usually the solution of an integral equation itself that is of interest, but certain integrals of it or the derivatives of the integrals. To improve the accuracy, it is therefore advisable to introduce the approximation in question, especially as the differential properties of the solution are (as a rule) known.

10. Method of Successive Approximations

In Sec. 1 an account has been given of the analytic theory of a resolvent according to which the solution of a Fredholm integral equation of the second kind

$$\varphi(x) = \lambda \int_a^b K(x, y) \varphi(y) dy + f(x) \tag{10.1}$$

can be constructed as a series (for certain values of λ)

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(x), \tag{10.2}$$

the terms of this series being determined by the following recurrence relations:

$$\varphi_n(x) = \int_a^b K(x, y) \varphi_{n-1}(y) dy \quad (n = 1, 2, \ldots), \quad \varphi_0(x) = f(x). \tag{10.3}$$

The calculations present no great difficulties if we follow certain quadrature formulas. We begin with the simplest formula, viz. the rectangular formula.

Let the segment $[a, b]$* be divided by the points x_i $(i = 1, 2, \ldots, \ldots, m+1,\ x_1 = a,\ x_{m+1} = b)$ into m parts. These points will be further referred to as nodal points. We introduce into consideration points x_i' situated in the central part of each elementary segment and called pivotal points.

In accordance with relations (10.3) we determine the function $\varphi_0(x)$ at the nodal and pivotal points and then calculate the value of the function $\varphi_1(x_j')$ at all pivotal points with the aid of the rectangular quadrature formula:

$$\varphi_1(x_j') \approx \sum_{i=1}^{m} K(x_j', x_i') \varphi_0(x_i') \Delta x_i'. \tag{10.4}$$

Here $\Delta x_i'$ is the length of the ith segment $(x_{i+1} - x_i)$. By repetition of the above procedure we find all subsequent terms of series (10.2). The disadvantage of this scheme (apart from its poorest accuracy in comparison with other formulas) is the necessity for modification in the case when the kernel $K(x, y)$ is unbounded on the diagonal $x = y$. In the latter case it is necessary to omit the jth term in the sum, and this involves an additional significant error when the size of the elementary segments is large.

To improve the accuracy and to overcome the difficulties associated withsingularities,we may also use the mixed quadrature formula when

* For simplicity, we take a one-dimensional case.

the values of the function $\varphi_1(x)$ are assumed, as before, to be constant within a segment, and for the kernel we take the mean value with respect to the nodal points. We obtain

$$\varphi_1(x_j') \approx \frac{1}{2} \sum_{i=1}^{m} \{K(x_j', x_{i+1}) + K(x_j', x_i)\} \varphi_0(x_i') \Delta x_i', \tag{10.5}$$

and similarly for the subsequent terms of series (10.2).

In the work of P. I. Perlin [10] a method is proposed in which the trapezoidal formula is applied to the whole integrand. In the first stage this involves no difficulty since the values of the function $\varphi_0(x)$ are known at the nodal points. In the second stage, however, the determination of the function $\varphi_2(x)$ requires that the values of the function $\varphi_1(x)$ should first be found at the nodal points. These values are found by interpolation from the values of the function $\varphi_1(x)$ at the nearest pivotal points (for example, assuming $\varphi_1(x_i) = \frac{1}{2}[\varphi_1(x_j') + \varphi_1(x_{j-1}')]$). Further operations are obvious. The corresponding sum then takes the form

$$\varphi_1(x_j') \approx \frac{1}{2} \sum \{K(x_j', x_{i+1}) \varphi_0(x_{i+1}) + K(x_j', x_i) \varphi_0(x_i)\} \Delta x_i'. \tag{10.6}$$

The same accuracy can be achieved by using a double subdivision of the segment $[a, b]$, i.e., by performing the calculations once again with the points x_i as pivotal points and x_j as nodal points. But this procedure will naturally lead to an unjustifiable increase in the amount of numerical work involved.

We now turn to the proof of the convergence of the schemes described above and those similar to them. In the literature (L. V. Kantorovich, V. I. Krylov [1]) there are corresponding estimates available expressed in terms of the derivatives of a kernel. These estimates allow the proof of the convergence of computational schemes,* but for most integral equations encountered in the theory of elasticity these results cannot be directly used because of the singularity of kernels and also because of the fact that the given values of λ are either eigenvalues or coincide with them in modulus.

The general procedure for proving the convergence of computational schemes for the method of successive approximations is as follows. Let us fix the number of terms in series (10.2). Since a finite number of integrals must be evaluated for the corresponding sum, this sum can be calculated with any preassigned accuracy by using an increasingly finer division. If the number of terms in the series is increased, it is necessary to decrease the size of the segments. True, this proof does not take into account any possible magnification of computational errors.

* Naturally, in the case of the convergence of the exact algorithm.

From the discussion in Sec. 1 it follows that series (10.2) is absolutely convergent when the corresponding value of λ is less in modulus than the first eigenvalue. Consider the case when λ coincides in modulus with the first eigenvalue, but is not equal to it. Let the value λ_0 of interest be taken equal to unity, and let the eigenvalue be first placed at the point -1. Naturally, series (10.2) is divergent. But, as mentioned in Sec. 1, the resolvent [actually constructed by series (10.2)] is a function analytic in λ outside its poles. To obtain a convergent representation at the point $\lambda = 1$, it might be well, therefore, to use the method of analytic continuation. We take a point λ_1, situated between 0 and 1, and obtain a solution at this point according to (10.2). To construct the solution of interest to us at the point $\lambda = 1$, this series must be reexpanded in an argument $\lambda' = 1 - \lambda_1$, substituting this argument in the final representation. The reconstructed series is conditionally convergent and is of the form (if $\lambda_1 = 0.5$)

$$\begin{aligned} \varphi(x) &= \varphi_0(x) + \left(\lambda' + \frac{1}{2}\right)\varphi_1(x) + \left(\lambda' + \frac{1}{2}\right)^2 \varphi_2(x) + \ldots = \\ &= \left[\varphi_0(x) + \frac{1}{2}\varphi_1(x) + \frac{1}{4}\varphi_2(x) + \frac{1}{8}\varphi_3(x) + \ldots\right] + \\ &\quad + \left[\varphi_1(x) + \varphi_2(x) + \frac{3}{4}\varphi_3(x) + \ldots\right]\lambda' + \\ &\quad + \left[\varphi_2(x) + \frac{3}{2}\varphi_3(x) + \ldots\right]\lambda'^2 + \ldots \end{aligned} \tag{10.7}$$

It is essential to note that the functions $\varphi_n(x)$ are found by means of the same recurrence relations (10.3) which are entirely independent of the values of λ. Under the adopted conditions on the disposition of the values of λ, the intermediate point can be taken on the real axis [as was done in formula (10.7)].

Let us now consider the so-called method of pole elimination by remultiplication assuming, in addition, that the point $\lambda = -1$ is a simple pole of the resolvent. In this case the function $(\lambda + 1)\,\Gamma(x, y, \lambda)$ and hence the function $(\lambda + 1)\,\varphi(x)$ have no pole at the point $\lambda = -1$. The series for the functions $(\lambda + 1)\,\Gamma(x, y, \lambda)$ and $(\lambda + 1)\,\varphi(x)$ are therefore convergent in a circle centred at zero up to the second eigenvalue, which is assumed to be greater than unity in modulus.

It is obvious that the series for the function $(\lambda + 1)\,\varphi(x)$ is of the form

$$(\lambda + 1)\,\varphi(x) = \varphi_0(x) + \sum_{n=1}^{\infty} \lambda^n [\varphi_n(x) + \varphi_{n-1}(x)] \tag{10.8}$$

and for $\lambda = 1$ we obtain

$$\varphi(x) = 0.5\left\{\varphi_0(x) + \sum_{n=1}^{\infty} [\varphi_n(x) + \varphi_{n-1}(x)]\right\}. \tag{10.9}$$

With the purpose of increasing the effectiveness of the method of successive approximations, certain recommendations have been developed to transform series (10.2) by a change of the variable $\lambda = \omega(\eta)$, and in particular by conformal mapping. Some recommendations of this kind may be found in the literature (see L. V. Kantorovich, V. I. Krylov [1] and V. N. Kublanovskaya [1]). Naturally, the change of the variable is dictated by the information on the position of poles of the resolvent.

We now proceed to the case when the given value λ_0 is a simple pole of the resolvent. It has been shown in Sec. 1 that the factor multiplying the highest* power in the series expansion of the resolvent is an eigenfunction of the companion equation (as a function of the second argument). In fulfilling the condition for the orthogonality of the right-hand side of the original equation to the eigenfunction of the companion equation, the representation of the solution in terms of the resolvent can therefore be simplified by rejecting the above term of the series, and this leads to the proof that the solution of the equation is an analytic function in λ in a circle of radius larger than the modulus $|\lambda_0|$. Consequently, here, too, the method of successive approximations must lead to the solution. This conclusion is valid, however, only in the case when all calculations are carried out according to (10.3) with an absolute accuracy, which is possible only in the simplest cases, far from always of interest. The error in calculations necessarily entails the violation of conditions (1.38), with the exception of elementary cases, which will be discussed below.

In the work of B. Aliev [1] the solution of integral equations of the second kind (with symmetric kernels) on the spectrum is considered in terms of the theory of incorrect problems (see A. N. Tikhonov, V. Ya. Arsenin [1]). It is assumed that all calculations are carried out accurately, but the right-hand side $f(x)$ of the original equation is given with a certain error δ. Series (10.2) in this case should be understood in a special sense (let it be called asymptotic).

The algorithm is convergent if $\delta^2 N$ tends uniformly to zero, where N is the number of terms retained in the series expansion (10.2). It follows from this estimate that we cannot retain an arbitrarily large number of terms in the series expansion for a fixed value of δ.

In reference to the problem under consideration concerning the influence of computational error on the solution of the integral equation the above result apparently leads to the similar conclusion that series (10.2) should be understood as asymptotic. To obtain the solution, proceed as follows. Assign the number of terms N retained in the expansion and choose so fine a division of the segment as to achieve the specified accuracy for the required finite sum. If the

* The first in this case.

computational error is disregarded,* an arbitrarily assigned accuracy can always be obtained since a finite number of integrals is evaluated with a desired accuracy. As the number of terms retained in the series expansion is increased, the size of elementary sections must accordingly be reduced. Ultimately it is possible to determine, with a preassigned accuracy, the values of several finite sums, and if the difference between the last two sums is within this accuracy, the calculations must be stopped.

Consider, now, another procedure of constructing a convergent representation (see P. I. Perlin [10]). Let $\psi_1(x)$ be a normalized eigenfunction of the companion equation

$$\psi(x) = \lambda \int_a^b K(y, x)\, \psi(y)\, dy. \tag{10.10}$$

We transform the right-hand side of Eq. (10.1) by adding the expression

$$-\psi_1(x) \int_a^b f(y)\, \psi_1(y)\, dy. \tag{10.11}$$

In general, this addition must vanish by virtue of condition (1.38), but if it is understood according to any quadrature formula, it is a small quantity. The advantage of the right-hand side of this form lies in the fact that it is strictly orthogonal to the function $\psi_1(x)$ (according to the same quadrature formula):

$$\int_a^b \left\{ f(x) - \psi_1(x) \int_a^b f(y)\, \psi_1(y)\, dy \right\} \psi_1(x)\, dx =$$

$$= \int_a^b f(x)\, \psi_1(x)\, dx - \int_a^b \psi_1^2(x)\, dx \int_a^b f(y) \psi_1(y)\, dy = 0. \tag{10.12}$$

Note that this equality holds when the normalization of the function $\psi_1(x)$ is performed by the same quadrature formula. Since each function $\psi_n(x)$ must be orthogonal to the right-hand side (see Sec. 1), it is natural, in passing to each new function $\varphi_n(x)$, to make a correction similar to (10.11)

$$\tilde{\varphi}_n(x) = \varphi_n(x) - \psi_1(x) \int_a^b \varphi_n(y)\, \psi_1(y)\, dy \quad (n = 1, 2, \ldots), \tag{10.13}$$

and use the functions $\tilde{\varphi}_n(x)$ to construct $\varphi_{n+1}(x)$.

* The error inherent in quadrature formulas remains.

The foregoing leads to a rigorous proof of convergence if by the function $\psi_1(x)$ is meant not the eigenfunction of the exact companion equation, but the eigenfunction of a companion equation to the approximate integral equation actually being constructed.

It is natural that for systems of integral equations with a simple pole of the resolvent there are several eigenfunctions of the companion equation and so the additions are constructed for each of them. It is essential that in this case the eigenfunctions must be taken in orthonormal form, the orthonormalization being performed by the quadrature formula used to form additions.

Note that some special procedures of constructing convergent algorithms will be described in studying specific equations (Secs. 18, 19).

It should be emphasized that if an integral equation is solved on the boundary of a circle of convergence (but not on the spectrum) and if the right-hand side is orthogonal to the eigenfunction of the companion equation (for the eigenvalue), there is no need to use the various procedures described above (elimination of a pole by remultiplication and similar procedures) since series (10.2) itself is convergent.

We shall mention the work of S. G. Mikhlin [6] where the problem of the approximate construction of a resolvent is solved by replacing the kernel by a degenerate one and estimates of errors are obtained.

It has been assumed above that the integral equations are regular. Let us now briefly discuss the special features involved in solving singular equations by successive approximations. It is obvious that if the convergence of the method is proved, its implementation reduces to evaluating the integrals according to the recurrence relations (10.3), and the presence of a singularity only necessitates the use of special quadrature formulas.

In the work of S. G. Mikhlin [2] (see Sec. 8) it is proved that if the norm of the singular operator (determined by the norm of the regularizing operator) is sufficiently small, the method of successive approximations is convergent. V. D. Kupradze [3] justified the application of the method for solving two-dimensional singular integral equations of the theory of elasticity when the medium is piecewise homogeneous [in the special case of the so-called principal contact problem (Sec. 36)], and Pham The Lai [1] extended this result to the case of the (first and second) fundamental three-dimensional problems of the theory of elasticity (Sec. 33). Some questions concerning the convergence of the method of successive approximations in reference to equations arising in the solution of elasticity problems are further discussed in Sec. 33.

The construction of quadrature formulas for singular integrals in the one-dimensional case will be considered in Sec. 12. In the case of two and more dimensions the quadrature formulas are obtained only for kernels of a special kind. In the work of B. G. Gabdulkhaev

[2] a singular integral with the Hilbert kernel is considered, namely

$$\int_0^{2\sigma}\int_0^{2\pi} \cot\frac{x-x'}{2}\cot\frac{y-y'}{2}\,\varphi(x,\ y)\,dx\,dy,$$

and in the work of A. I. Vaindiner and V. V. Moskvitin [1] an integral with kernels appearing in integral equations of the theory of elasticity (see also Secs. 31, 33).

11. Mechanical Quadrature Method for Regular Integral Equations

Consider again the Fredholm integral equation (10.1), which can conveniently be written in an alternate form:

$$x(s)-\lambda\int_0^1 H(s,\ t)\,x(t)\,dt=y(s).$$

We use the division of a segment into m parts introduced in the preceding section. Any one of the methods, proposed and mentioned above, for the approximate evaluation of integrals at the pivotal points with arbitrarily assigned values of the density function leads to a system of linear equations in these values. The available estimates make it possible to prove (under certain restrictions on the kernel and the right-hand side) the convergence of the network representation of the solution thus obtained to the exact one. Moreover, we have the following fundamental results.

If the integral equation is solvable,* so is the corresponding linear system, at least beginning with a fairly fine division.

The proof consists in obtaining certain estimates necessary for the application of the general principles of the theory of approximate methods (see Sec. 9).

The quadrature formula for an arbitrary function is represented as

$$\int_0^1 x(t)\,dt=\sum_{k=1}^{n} A_k^{(n)}\,x\left(t_k^{(n)}\right). \tag{11.1}$$

The points t_k are also supplied with a superscript to allow thereby for the variation of their total number and arrangement. The equation is then replaced by the following system:

$$\begin{aligned} x\left(t_i^{(n)}\right)-\lambda\sum_{k=1}^{n} A_k^{(n)}\,H\left(t_i^{(n)},\ t_k^{(k)}\right)x\left(t_k^{(n)}\right)&= \\ =y\left(t_i^{(n)}\right)\quad (i=1,\ 2,\ldots,\ n).& \end{aligned} \tag{11.2}$$

* The given value of λ is not an eigenvalue.

Note that when the segment is divided into n equal sections and the rectangular quadrature formula is used, we have

$$t_k^{(n)} = \frac{2k-1}{2n}, \quad A_k^{(n)} = \frac{1}{n} \quad (k = 1, 2, \ldots, n).$$

For simplicity, we assume that the functions $H(s, t)$ and $y(s)$ and hence $x(s)$ are periodic functions with period 1. The space X is taken to be the space C of continuous periodic functions (with period 1). For the space X' we take the set of broken periodic functions with the abscissas of vertices at the points $t_k^{(n)}$. Let $\overline{X}$ be a finite-dimensional space m_n consisting of elements $\overline{x} = (\xi_1, \xi_2, \ldots, \xi_n)$ with the norm $\|x\| = \max |\xi_i|$, and let the image of the function be its values at n points:

$$\overline{x} = \varphi x = [x(t_1^{(n)}), \ldots, x(t_n^{(n)})].$$

Let $x = (\eta_1, \eta_2, \ldots, \eta_n)$, then the element $x' = \varphi_0^{-1} x$ (of the space X') is a piecewise linear function with vertices at the points $(t_k^{(n)}, \eta_k)$. We have the obvious equalities:

$$\|\varphi\| = \|\varphi_0^{-1}\| = 1.$$

Denote by $\overline{H}$ an operator in the space $\overline{X}$ specified by the matrix $\| A_k^{(n)} H(t_i^{(n)}, t_k^{(n)}) \|$. Referring to Sec. 9, we conclude that to prove Theorem I, we must only check the fulfilment of conditions (9.3), (9.5), (9.16).

Denote by $\omega^s(\delta)$ and $\omega^t(\delta)$ the moduli of continuity of the kernel with respect to each of the arguments and proceed to estimate the norm:

$$\|\varphi H x' - \overline{H} \varphi x'\| = \max_i \left| \int_0^1 H(t_i, t)\, x'(t)\, dt - \right.$$

$$\left. - \sum_{k=1}^{n} A_k H(t_i, t_k)\, x'(t_k) \right| =$$

$$= \max_i \left| \sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} H(t_i, t)\, x'(t)\, dt - \right.$$

$$- \sum_{k=1}^{n} H\left(t_i, \frac{t_k + t_{k+1}}{2}\right) x'(t)\, dt +$$

$$+ \sum_{l=1}^{n} \frac{1}{2n} H\left(t_i, \frac{t_k + t_{k+1}}{2}\right) [x'(t_k) + x'(t_{k+1})] -$$

$$- \sum_{k=1}^{n} \frac{1}{n} H(t_i, t_k)\, x'(t_k) \Big| \leqslant$$

$$\leqslant \left[\omega^t\left(\frac{1}{2n}\right) + \frac{1}{2}\omega^t\left(\frac{1}{2n}\right) + \frac{1}{2}\omega^t\left(\frac{1}{2n}\right) \right] \| x' \| =$$

$$= 2\omega^t\left(\frac{1}{2n}\right) \| x' \|. \qquad (11.3)$$

Consequently, condition (9.3) is fulfilled, with $\varepsilon = 2\,\omega^t\,(1/2n)$.

Let us now check condition (9.5). For the function $x' \in X'$ approximating the integrals

$$g(s) = \int_0^1 H(s, t)\, x(t)\, dt,$$

we take a piecewise linear function coinciding with them at the points t_i, i.e., for $t_i \leqslant s \leqslant t_{i+1}$:

$$x'(s) = n\,(t_{i+1} - s) \int_0^1 H(t_i, t)\, x(t)\, dt + n\,(s - t_i) \int_0^1 H(t_{i+1}, t)\, x(t)\, dt.$$

We obtain the estimate

$$|g(s) - x'(s)| = \Big| \int_0^1 \{H(s, t) - n\,(t_{i+1} - s)\, H(t_i, t) -$$

$$- n\,(s - t_i)\, H(t_{i+1}, t)\}\, x(t)\, dt \Big|.$$

Since the expression in the braces is estimated by $\omega^s\,(1/n)$, we obtain

$$\| Hx - x' \| \leqslant \omega^s\left(\frac{1}{n}\right) \| x \|, \qquad (11.4)$$

and hence $\varepsilon_1 = \omega^s\,(1/n)$.

A similar argument for the function y leads to the estimate

$$\| y - y' \| \leqslant \omega^y\left(\frac{1}{n}\right), \qquad (11.5)$$

where $\omega^y\,(\delta)$ is the modulus of continuity of the function y. Hence, condition (9.16) is fulfilled, and

$$\varepsilon_2 = \omega^y\left(\frac{1}{n}\right) \frac{1}{\| y \|}. \qquad (11.6)$$

Thus, the constant q appearing in Theorem I and taking the form*

$$q = |\lambda| \left[\| K \|\, \omega^s\left(\frac{1}{n}\right) + 2\omega^t\left(\frac{1}{2n}\right) + 2|\lambda|\,\omega^s\left(\frac{1}{n}\right) \omega^t\left(\frac{1}{2n}\right) \right] \cdot \| K^{-1} \|,$$

* It will be recalled that $Kx = x - \lambda Hx$.

can be made less than unity by suitably increasing n, which leads to the required result.

We give the final formulas for the estimation of the error of the approximate solution $\varphi_0^{-1}x_0$:

$$\| x^* - \varphi_0^{-1} x_0 \| \leqslant \left\{ 4 |\lambda| \omega^t \left(\frac{1}{2n}\right) \| \overline{K}^{-1} \| + \right.$$

$$\left. + \left(|\lambda| \omega^s \left(\frac{1}{n}\right) + \frac{\| K \| \omega_y \left(\frac{1}{n}\right)}{\| y \|} \right) (1 + \| K \| \cdot \| \overline{K}^{-1} \|) \right\} \| x^* \|.$$

Similar results can be obtained for the non-periodic case if use is made of the trapezoidal formula.

We now turn to the consideration of integral equations when λ is an eigenvalue. The mechanical quadrature method can, of course, be formally applied here, but this involves great difficulties. First, the resulting system of linear equations is degenerate or nearly degenerate, and to ensure the stability of solution we must use regularization procedures (see A. N. Tikhonov, V. Ya. Arsenin [1]). Second, when stable values are obtained, it is required to prove that in the limit the solution of the system tends to the exact solution of the integral equation.

Note that in some cases special procedures can be used. For example, if the eigenfunctions of the original and companion equations are known, one should pass to Eq. (1.36), which no longer involves the eigenvalue. In Sec. 18, in discussing the question concerning the numerical implementation of Muskhelishvili's equation we describe a procedure due to P·I. Perlin and Yu. N. Shalyukhin [1].

12. Approximate Methods for Solving Singular Integral Equations

From the results obtained in Chap. I it follows that singular integral equations can be transformed to equivalent regular equations (in the case of multidimensional equations only the existence of such a transformation is proved). The solution of singular equations can therefore, in general, be obtained by passing to the corresponding regular equations and finding their approximate solution. The actual implementation of this procedure is, however, too laborious because of the considerable difficulty involved in constructing the required regular equation (both the kernel and the right-hand side).

It is more expeditious to construct the solution of singular (one-dimensional) integral equations directly.* For the most part, the

* The case of more than one dimension is not considered because of the lack of practically any general investigations. Special equations (equations of the theory of elasticity) are discussed in Sec. 36 and the following sections.

methods used are a generalization of the procedures employed for solving regular integral equations with due regard for the specific features of the evaluation of singular integrals. Essential use is made of the fact that the characteristic singular integral equations are solvable in explicit form.

The solution of singular integral equations (both the construction of approximate algorithms and their justification) is covered adequately in the literature. We shall mention the works of M. A. Lavrent'ev, S. G. Mikhlin, A. I. Kalandiya, V. V. Ivanov, B. G. Gabdulkhaev.

To begin with, we consider the cases of the straightforward solution of singular equations both for closed and unclosed contours without first transforming a contour into a unit circle γ or a segment L_1 $(-1, 1)$. For simplicity, we restrict ourselves to the case of a single contour.

Suppose (see S. G. Mikhlin, Kh. L. Smolitskii [1]) that the kernel of a regular operator is a degenerate function:

$$k(t, \tau) = \sum_{k=1}^{n} u_k(t)\, v_k(\tau). \tag{12.1}$$

The original singular equation may be represented, after transposing the regular term, as

$$a(t)\,\varphi(t) + \frac{b(t)}{\pi i} \int_L \frac{\varphi(\tau)\, d\tau}{\tau - t} = f(t) - \sum_{k=1}^{n} c_k u_k(t), \tag{12.2}$$

where

$$c_k = \int_L v_k(\tau)\, \varphi(\tau)\, d\tau.$$

The coefficients c_k are assumed to be arbitrarily given. The equation can then be solved explicitly (assuming that for a negative value of the index the function $f(t)$ satisfies the orthogonality conditions). The resulting solution of the characteristic equation depends on the parameters c_k, and the substitution of this solution in the equalities determining the constants c_k leads to a linear system.

We now turn to the question of evaluating singular integrals, which naturally precedes the consideration of the approximate solution of singular integral equations. If the density functions of singular integrals were known, the evaluation of the integrals would be an elementary operation because of the availability of the regular representations (2.5) and (2.18). Moreover, the integrals can be directly evaluated as improper ones by first dividing the contour into sections of equal length and excluding, according to the definition of a singular integral, two sections adjacent to the singular

point. Let t_i ($i = 1, 2, \ldots, n+2$, $t_1 = a$, $t_{n+2} = b$) be such a set of points. The lengths of all chords $t_{i+1} - t_i$ must be equal, with the possible exception of the length of the chord $t_{n+2} - t_{n+1}$, which must not be larger than the rest. The points of the curve L will also be determined by the arc length s measured from the end point $t_1 = a$. Let M denote $\max \left| \frac{s - s_0}{t - t_0} \right|$ $(t, t_0 \in L)$. We choose the number n so large as to fulfil the following inequalities on the arc (t_j, t_{j+1}):

$$| t_{j+1} - t_j | \leqslant | t_k - t_j | \quad (k \neq j),$$

$$| s_j - s | \leqslant 2 \, | t_j - t | \leqslant 4 \, | t_{j+1} - t_j |.$$

V. V. Ivanov [1] has proved that the quadrature formula obtained by the proposed method [if the function $\varphi(t)$ is specified discretely at the points t_j], namely

$$\frac{1}{\pi i} \int_L \frac{\varphi(t)}{t - t_k} dt = \frac{1}{\pi i} \ln \frac{b - t_k}{t_k - a} - \\ - \sum_{j=1}^{n+1}{}' \, \frac{\varphi(t_j) - \varphi(t_k)}{t_j - t_k} \frac{1}{\pi i} (t_{j+1} - t_j), \tag{12.3}$$

where a prime indicates that the summation is carried out over all $j \neq k$, $k - 1$ ($k = 1, 2, \ldots, n + 1$), has the error

$$\Delta n < \frac{8}{\pi} A \left(\frac{1}{\alpha} + M \ln 2n \right) \left(\frac{S}{n} \right)^{\alpha}.$$

Here S is the total arc length, A is the constant in the H-L condition.

We now proceed to the case when the density function has a singularity, say, at the point $t_1 = a = -1$. We restrict ourselves to the case of the contour L_1. Denote by $\sigma_n(t_0)$ a neighbourhood of the point t_0 centred at it and of length $2/n$, and take n so large that $| t_0 - a | > 3/n$. We next divide the contour L_1 into $2n$ equal parts by points $t_1, t_2, \ldots, t_{2n+1}$. The quadrature formula is then of the form

$$\int_L \frac{\varphi(\tau) \, d\tau}{(\tau + 1)^{\gamma} (\tau - 1)} = \varphi(-1) \int_{L_1} \frac{d\tau}{(\tau + 1)^{\gamma} (\tau - t)} + \\ + \sum_{j=1}^{2n}{}' \, \frac{\varphi(t_j) - \varphi(-1)}{t_j + 1} \, \frac{(t_{j+1} + 1)^{1-\gamma} - (t_j + 1)^{1-\gamma}}{1 - \gamma}. \tag{12.4}$$

The integral appearing in it is represented by the series

$$\int_{L_1} \frac{d\tau}{(\tau + 1)^{\gamma} (\tau - t)} = \frac{\pi \cot j\pi}{(t + 1)^{\gamma}} + \sum_{s=0}^{\infty} \frac{1}{(1 + \gamma)^{2s+\gamma}} (t + 1)^s. \tag{12.5}$$

A prime on the summation indicates that the sum is carried out over all j such that the segment $(t_{j+1}, t_j) \notin \sigma_n(t)$. If the singularity is at the point 1, it is necessary to make a change of signs in some terms. The error in the quadrature formula is estimated as

$$\Delta n < \frac{6}{|t+1|^{\gamma}}\left[\frac{1}{\alpha}+\frac{1}{1-\lambda}+\ln\frac{n^2(1-t^2)}{2}\right]\frac{1}{n^{\alpha}}$$
$$(0 \leqslant \lambda = \operatorname{Re}\gamma < 1).$$

Note that if in constructing the quadrature formulas we proceed from the polygonal approximation of the density function, i.e., we replace $\varphi(t)$ by linear functions

$$\varphi^*(t) = \varphi(t_k)\frac{t_{k+1}-t}{t_{k+1}-t_k} + \varphi(t_{k+1})\frac{t-t_k}{t_{k+1}-t_k} \quad (t \in (t_k, t_{k+1})),$$

the order of accuracy is improved. For a closed contour, the estimate of the error is (see A. A. Korneichuk [1]) $C \ln n \cdot n^{-2}$.

V. V. Ivanov [1] used Faber's polynomials $\Phi_k(z)$ for constructing quadrature formulas. (Faber's polynomial of order k is a polynomial part of the kth power of a function effecting the conformal mapping of the outside of the contour into the outside of a circle and having the highest coefficient equal to unity.) In this case the expression for the singular integral is of the form

$$\frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}\,d\tau = -\varphi(t) + \sum_{k=1}^{\infty} \varphi_k \Phi_k(t), \tag{12.6}$$

where

$$\varphi_k = \frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\Phi^k(\tau)}\Phi'(\tau)\,d\tau.$$

If only n terms are retained in representation (12.6), the error in the resulting formula is $Cn^{\varepsilon-\alpha}$, where α is the index of the class H-L to which the function $\varphi(t)$ belongs, and $\varepsilon > 0$. If, however, the function $\varphi(t)$ has a derivative of order p belonging to the class H-L, the estimate becomes stronger, viz. $Cn^{\varepsilon-p-\alpha}$.

Let now L be a smooth unclosed contour with the ends a and b. Suppose that the density function is representable as $\frac{\varphi(t)}{(t-a)^{\gamma}}$ (Re $\gamma <$ < 1). The corresponding singular integral is then

$$\frac{1}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{(\tau-a)^{\gamma}(\tau-t)} = \frac{\varphi(a)\cot\gamma\pi}{\pi i(\tau-a)^{\gamma}} +$$
$$+\frac{\varphi(t)}{\pi i}\sum_{k=0}^{\infty}\frac{(b-a)^{-\gamma-k}}{\gamma+k}(t-a)^k + \sum_{k=1}^{\infty}\varphi_k\frac{1}{2\pi i}\int_L \frac{\Phi_k(\tau)-\Phi_k(t)}{(\tau-a)^{\gamma}(\tau-t)}\,d\tau. \tag{12.7}$$

A similar formula (apart from signs) can be obtained for the case when the density function has a singularity at the end b. It is appropriate to mention here that Faber's polynomials for the segment L_1 coincide with Chebyshev's polynomials.

We next consider some special cases of the representation of the density function where the evaluation of singular integrals is elementary. Let γ be, as before, a circumference of unit radius. We take the function $\varphi_n(t)$ as a segment of the Fourier series:

$$\varphi_n'(t)=\sum_{k=-n}^{n} a_k t^k. \tag{12.8}$$

The value of the singular integral of the finite sum (12.8) is then

$$\frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\tau)}{\tau-t}\,d\tau=\sum_{k=0}^{n} a_k t^k-\sum_{k=-n}^{-1} a_k t^k. \tag{12.9}$$

If the series representation is constructed using the values of the density function at the points $t_j=e^{i\theta_j}\left(\theta_j=\frac{2\pi}{2n+1}j\right)$, the approximating polynomial is of the form

$$\varphi_n(t)=\frac{1}{2n+1}\sum_{j=-n}^{n}\varphi(t_j)\left(\frac{t}{t_j}\right)^{-n}\frac{1-\left(\frac{t}{t_j}\right)^{2n+1}}{1-\left(\frac{t}{t_j}\right)}. \tag{12.10}$$

The singular integral itself is

$$\frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\tau)}{\tau-t}\,d\tau=$$

$$=\frac{1}{2n+1}\sum_{j=-n}^{n}\varphi(t_j)\left[1+\frac{2i\sin n\,\frac{\theta-\theta_j}{2}\sin(n+1)\,\frac{\theta-\theta_j}{2}}{\sin\frac{\theta-\theta_j}{2}}\right]. \tag{12.11}$$

We now proceed to the construction of the approximate solution of singular equations when the contour L is a circumference. Assume that the index $\varkappa\geqslant 0$, and hence the solution always exists. For definiteness, we seek a solution for which the Cauchy-type integral

$$\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau-z}\,d\tau$$

has the highest-order zero at infinity. We therefore choose the following series representation for the solution:

$$\tilde{\varphi}(t)=\sum_{k=0}^{n}\alpha_k t^k+\sum_{k=-n}^{-1}\alpha_k t^{k-\varkappa}. \tag{12.12}$$

The coefficients α_k can be determined in different ways (as in the case of regular equations). We may use the collocation (point matching) method by equating both sides of the equations at a certain set of points on the contour (for example, at the same points t_j). We may also require that the difference (in the mean) between the left-hand and right-hand sides should be minimum (the method of least squares, see S. G. Mikhlin [4]), etc.

In the work of B. G. Gabdulkhaev [1] the collocation method has been studied from the general standpoint of the theory of approximate methods (see Sec. 9). It is proved that when the index is zero and when the equation has no eigenfunctions, there is a value of n* such that the system obtained by the collocation method is always solvable. The approximate solution $\tilde{\varphi}(t)$ thus determined converges to the exact one with the rapidity

$$\|\varphi-\tilde{\varphi}\| \leqslant (A_2 \ln n + B_2)\, n^{-r-\alpha+\beta}, \tag{12.13}$$

where A_1, A_2, B_1, and B_2 are constants independent of n, β is an arbitrary constant, $\alpha > \beta > 0$. Inequality (12.13) is understood as the norm

$$\|\varphi\| = \max_{\gamma} |\varphi(t)| + \sup \frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^{\beta}}.$$

The method considered here has been studied by V. V. Ivanov [1] in a different functional space, and so he obtained another estimate.

We now turn to the direct solution of the singular equation on the circumference γ by the mechanical quadrature method (see Phan-van-Hap [1]). The equation is rewritten as

$$\varphi(t) + \frac{b(t)}{\pi i} \int\limits_{\gamma} \frac{\varphi(\tau)-\varphi(t)}{\tau - t}\, d\tau + \int\limits_{\gamma} k(t, \tau)\, \varphi(\tau)\, d\tau = f(t). \tag{12.14}$$

In Eq. (12.14) the normalization has already been performed, and hence $a(t) = 1$. As regards the kernel $k(t, \tau)$, we may assume the presence of an integrable singularity, i.e.,

$$|k(t, \tau)| < C\, |t-\tau|^{\mu-1} \quad (\mu > 0).$$

We next consider a functional equation resulting from the application of the quadrature formulas to the original equation:

$$\varphi(t) + \frac{b(t)}{\pi i} \sum_{k=1}^{n}{}' \frac{\varphi(t_k)-\varphi(t)}{t_k - t}\, \Delta t_k + \sum_{k=1}^{n}{}' k(t, t_k)\, \varphi(t_k)\, \Delta t_k = f(t), \tag{12.15}$$
$$t \in (t_j, t_{j+1}), \quad k < j-1, \quad k > j+1, \quad \Delta t_k = t_{k+1} - t_k.$$

* Namely, the one satisfying the restriction

$$(A_1 \ln r + B_1)\, n^{-r-\alpha+\beta} < 1.$$

As before, the points t_k divide the circumference into n equal parts. The functions $b(t)$, $f(t)$, $k(t, \tau)$ belong to the class H-L [except the points $\tau = t$ for the kernel $k(t, \tau)$]. To obtain the approximate solution of (12.15), we equate the left-hand and right-hand sides of this equation at the points t_k. We arrive at the system

$$\varphi(t_l) + \frac{b(t_l)}{\pi i} \sum_{\substack{k=1 \\ k<l-1,\, k>l+1}}^{n}{}' \frac{\varphi(t_k) - \varphi(t_l)}{t_k - t_l} \Delta t_k + \\ + \sum_{\substack{k=1 \\ k<l-1,\, k>l+1}}^{n}{}' k_{lk} \varphi(t_k) = f(t_l). \quad (12.16)$$

13. Approximate Methods for Solving Singular Integral Equations (Continued)

We now turn to the consideration of cases where the contour is discontinuous. From the discussion in Sec. 6 it follows that an equation with a discontinuous contour can always be reduced to an equation with a continuous contour. However, the equation becomes too unwieldy and, moreover, the contour of integration increases, which makes the efficient solution difficult with the same computer capacity.

In this case, too, the use of formula (12.4) and similar formulas makes it possible to obtain the solution of the integral equation. It is, of course, advisable (but not necessary) to include factors corresponding to singularities of the solution at the end points in the representation of the solution. These singularities are determined at once from the consideration of the auxiliary Riemann problem (see Sec. 5).

Much attention is given to the consideration of singular equations with constant coefficients $a(t)$ and $b(t)$ when the contour L is a segment designated as L_1. Equations of this type are frequently encountered in applications (see Sec. 23), and a variety of special techniques have been devised for their solution (M. A. Lavrent'ev, A. I. Kalandiya, G. N. Pykhteev and others).

The equation in question is

$$a\varphi(t) + \frac{b}{\pi i} \int_{L_1} \frac{\varphi(\tau)}{\tau - t} d\tau + \int_{L_1} k(t, \tau) \varphi(\tau) d\tau = f(t). \quad (13.1)$$

It appears from what has been proved in Sec. 6 that the solution of Eq. (13.1) is represented as

$$\varphi(t) = g(t)(1-t)^{\alpha}(t+1)^{\beta}, \quad (13.2)$$

where the function $g(t)$ is continuous and different from zero everywhere on the contour L_1, and the exponents α and β are determined by the equalities

$$\alpha = \frac{1}{2\pi i} \ln\left(\frac{a-ib}{a+ib}\right) + N, \quad \beta = -\frac{1}{2\pi i} \ln\left(\frac{a-ib}{a+ib}\right) + M. \quad (13.3)$$

The integers N and M are chosen so that the constants α and β will satisfy the restrictions $-1 < \operatorname{Re}\alpha$, $\operatorname{Re}\beta < 1$, which ensure the physical meaning of the solution of the original boundary value problem. It is obvious that the sum $\alpha + \beta$ is an integer equal to the index of the equation, and hence, when $\alpha + \beta = -1$ (the sum cannot be a smaller number), Eq. (13.1) is solvable if condition (4.13) is fulfilled. When $\alpha + \beta = 1$, to secure uniqueness of the solution, we must require the fulfilment of the condition

$$\int_{L_1} \varphi(\tau)\, d\tau = A, \quad (13.4)$$

where A is a constant.

The representation of the density function in the form of (13.2) suggests that the new function $g(t)$ should be sought in the form of a series in Jacobi polynomials (see, for example, D. Jackson [1]):

$$g(t) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(t). \quad (13.5)$$

It will be recalled that Jacobi polynomials are polynomials of the form

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}[(1-t)^{\alpha+n}(1+t)^{\beta+n}]. \quad (13.6)$$

When $\alpha = \beta = 0$, Jacobi polynomials coincide with Legendre polynomials, and when $\alpha = \beta = -1/2$ or $\alpha = \beta = 1/2$, Jacobi polynomials coincide with Chebyshev polynomials of the first and second kind, respectively.

The Jacobi polynomials (with fixed values of the parameters α and β) are an orthogonal system of functions on the segment L_1 with the weight

$$\omega(t) = (1-t)^{\alpha}(1+t)^{\beta},$$

and for the system to be orthonormal, we must further introduce the factors

$$\left[\frac{\alpha+\beta+2n+1}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(\alpha+\beta+n+1)\, n!}{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}\right]^{1/2} = \frac{1}{[\theta_n^{(\alpha,\beta)}]^{1/2}}.$$

In our case when the parameters α and β are not arbitrary (their sum k is an integer, 1, 0 or -1), we have the formula (see F. Erdogan,

G. D. Gupta, T. S. Cook [1])

$$\frac{1}{\pi}\int\limits_{L_1}\omega(\tau)\,P_n^{(\alpha,\beta)}(\tau)\,\frac{d\tau}{\tau-t}=$$

$$=-\frac{a}{b}\,\omega(t)\,P_n^{(\alpha,\beta)}(t)-2^{-k}\,\frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\pi}\,P_{n-k}^{(-\alpha,-\beta)}(t). \quad (13.7)$$

Equality (13.7) allows us to construct immediately an approximate method for solving the integral equation (13.1) if the solution is sought in the form of series (13.5). We arrive at the functional equation

$$\sum_{n=0}^{\infty}a_n\left[-\frac{2^{-k}b}{\sin\pi\alpha}P_{n-k}^{(-\alpha,-\beta)}(t)+h_n(t)\right]=f(t), \quad (13.8)$$

where

$$h_n(t)=\int\limits_{L_1}\omega(\tau)\,P_n^{(\alpha,\beta)}(\tau)\,k(t,\tau)\,d\tau.$$

To solve this equation, we may expand its left-hand and right-hand sides in a series of Jacobi polynomials $P^{(-\alpha,-\beta)}(t)$ $(l=0,1,\ldots)$ By equating the coefficients of like polynomials, we arrive at a system of algebraic equations:

$$-\frac{2^{-k}b}{\sin\pi\alpha}\theta_l(-\alpha,-\beta)\,a_{l+k}+\sum_{n=0}^{N}d_{nl}c_n=F_l \quad (l=0,1,\ldots,N), \quad (13.9)$$

where

$$d_{nl}=\int\limits_{L_1}P_l^{(-\alpha,-\beta)}(\tau)\,(1-\tau)^{-\alpha}\,(1+\tau)^{-\beta}h_n(\tau)\,d\tau,$$

$$F_l=\int\limits_{L_1}P_l^{(-\alpha,-\beta)}(\tau)\,(1-\tau)^{-\alpha}(1+\tau)^{-\beta}\,f(\tau)\,d\tau.$$

Let us analyse the solvability of this system, depending on the value of k. In the case of $k=-1$ Eq. (13.9) is solvable if the orthogonality condition is fulfilled; this condition may be written as

$$\int\limits_{L_1}\left[f(\tau)-\int\limits_{L_1}k(t,\tau)\,\varphi(\tau)\,d\tau\right]\frac{dt}{\omega(t)}=0. \quad (13.10)$$

It can be shown that the first equation of system (13.9) is equivalent to Eq. (13.10). Thus, we obtain a system of order $N+1$ for the $N+1$ unknowns a_n. When $k=0$, the system is uniquely solvable. When $k=1$, there are $N+2$ unknowns; hence, the solution of the system is not unique and recourse must be made to Eq. (13.4).

We now turn to the consideration of singular integral equations of a more general type than those studied previously:

$$\frac{1}{\pi}\int_{L_1}\frac{\varphi(\tau)}{\tau-t}\,d\tau+\frac{1}{\pi}\int_{L_1}k(t,\tau)\,\varphi(\tau)\,d\tau+\int_{L_1}k_1(t,\tau)\,\varphi(\tau)\,d\tau=f(t),\tag{13.11}$$

where the additional kernel $k_1(t, \tau)$ is of the form

$$\sum_{k=0}^{n} c_k(t+1)^k\frac{d^k}{dt^k}(\tau-z_1)^{-1}+\sum_{j=0}^{m} b_j(1-t)^j\frac{d^j}{dt^j}(\tau-z_2)^{-1}.$$

Here z_1 and z_2 stand, respectively, for the expressions $[(t+1)\,e^{i\theta_1}-1]$ and $[(t-1)\,e^{i\theta_2}+1]$, where θ_1 and θ_2 are constants in the range $0<\theta_1<2\pi$ and $-\pi<\theta_2<\pi$. Equations of this type arise, for example, in solving plane problems of the theory of elasticity involving external or interface cuts.

These equations differ from those studied in Chap. I in that they are not reducible to the Riemann boundary value problem; however, in this case, too, it may be stated that the solution is representable in the form

$$\varphi(t)=g(t)=(1-t)^{\alpha}(1+t)^{\beta}\quad(-1<\operatorname{Re}\alpha,\quad \operatorname{Re}\beta<1),$$

where $g(t)$ is a function belonging to the class H-L, and the constants α and β are determined from the analysis of the equation itself (using the properties of Cauchy-type integrals on unclosed contours).

We use the quadrature formula (A. H. Stroud, T. D. Secres [1]):

$$\int_{L_1}F(t,\tau)\,(1-\tau)^{\alpha}(1+\tau)^{\beta}\,d\tau\approx\sum_{k=1}^{N}W_kF(t_k,t),\tag{13.12}$$

where t_k are the roots of the equations

$$P_N^{(\alpha,\beta)}(t_k)=0,\quad(k=1,2,\ldots,N),$$

and the weight factors are

$$W_k=-\frac{2N+\alpha+\beta+2}{(N+1)!\,(N+\alpha+\beta+1)}\,\frac{\Gamma(N+\alpha+1)\,\Gamma(N+\beta+1)}{\Gamma(N+\alpha+\beta+1)}\times$$
$$\times\frac{2^{\alpha+\beta}}{P_N^{\prime(\alpha,\beta)}(t_k)\,P_{N+1}^{(\alpha,\beta)}(t_k)}.$$

The use of formula (13.12) for solving the integral equation by the collocation method leads to a system of algebraic equations:

$$\frac{1}{\pi}\sum_{k=1}^{N}g(t_k)\,W_k\left[\frac{1}{t_k-t_j^0}+K(t_j^0,t_k)+\pi k_1(t_j^0,t_k)\right]=f(t_j^0)\tag{13.13}$$
$$(j=1,2,\ldots,N-1).$$

The points t_j^0 are the roots of the equation

$$P_{N-1}^{(\alpha+1,\ \beta+1)}(t_j^0)=0,$$
$$-1<\operatorname{Re}\alpha,\quad \operatorname{Re}\beta<0\quad (j=1,\ \ldots,\ N-1).$$

With other restrictions on α and β the points t_j^0 are the roots of the equations

$$P_{N+1}^{(\alpha-1,\ \beta-1)}(t_j^0)=0\quad (0<\operatorname{Re}\alpha<1,\quad 0<\operatorname{Re}\beta<1),$$
$$P_{N}^{(\alpha-1,\ \beta+1)}(t_j^0)=0\quad (0<\operatorname{Re}\alpha<1,\quad -1<\operatorname{Re}\beta<0),$$
$$P_{N}^{(\alpha+1,\ \beta-1)}(t_j^0)=0\quad (-1<\operatorname{Re}\alpha<0,\quad 0<\operatorname{Re}\beta<1).$$

Note that if $-1<\operatorname{Re}\alpha$, $\operatorname{Re}\beta<0$, the solution of Eq. (13.13) demands recourse to equality (13.4) whose approximate implementation leads to a further relation

$$\sum_{k=1}^{N} g(t_k)\,W_k=A,$$

in addition to system (13.13).

We proceed to determine the values of α and β. Consider the Cauchy-type integral

$$\Phi(z)=\frac{1}{\pi}\int\limits_{L_1}\frac{\varphi(t)}{t-z}\,dt=\frac{1}{\pi}\int\limits_{L_1}\frac{g(t)(1-t)^{\alpha}(1+t)^{\beta}}{t-z}\,dt.$$

This integral admits the representation

$$\Phi(z)=-g(-1)\,2^{\alpha}\frac{e^{-i\pi\beta}}{\sin\pi\beta}(z+1)^{\beta}+$$
$$+g(1)\,2^{\beta}\frac{1}{\sin\pi\alpha}(z-1)^{\alpha}+\Phi_0(z),\quad (13.14)$$

where the function $\Phi_0(z)$ is bounded everywhere except possibly at the points -1 and 1 at which there can be singularities of order less than $\operatorname{Re}\alpha$ and $\operatorname{Re}\beta$. Further, based on the Sokhotskii-Plemelj formulas (2.9′), we obtain an equivalent representation for the singular integral:

$$\frac{1}{\pi}\int\limits_{L_1}\frac{\varphi(t)}{\tau-t}\,dt=$$
$$=-g(-1)\,2^{\alpha}\cot\pi\beta\,(t+1)^{\beta}+g(1)\,2^{\beta}\cot\pi\alpha\,(1-t)^{\alpha}+F_0(t),\quad (13.15)$$

where the behaviour of the function $F_0(t)$ is similar to that of the function $\Phi_0(z)$.

We next consider the integral

$$\frac{1}{\pi}\int_{L_1}\frac{\varphi(\tau)}{\tau-z_1}\,d\tau=\Phi(z_1).$$

It will be recalled that the points z_1 fall on a line (segment) when the points t run through the segment L_1. This integral admits the representation

$$\Phi(z_1)=-g(-1)\,2^{\alpha}\frac{e^{-\pi i\beta}}{\sin\pi\beta}e^{i\beta\theta_1}(t+1)^{\beta}+F_1(t). \qquad (13.16)$$

The properties of the function $F_1(t)$ in the neighbourhood of the point -1 are obvious.

Consider, now, the identity

$$\frac{1}{\pi}\int_{L_1}\varphi(\tau)(t+1)^k\frac{d^k}{dt^k}(\tau-z_1)\,d\tau=(t+1)^k\cdot\frac{d^k}{dt^k}\Phi(z_1),$$

which enables us to obtain the representation

$$\frac{1}{\pi}\int_{L_1}\varphi(\tau)(t+1)^k\frac{d^k}{dt^k}(\tau-z_1)d\tau=$$

$$=-g(-1)2^{\alpha}\frac{e^{-\pi i\beta}}{\sin\pi\beta}e^{i\beta\theta_1}\beta(\beta-1)\dots(\beta-k+1)(t+1)^{\beta}+$$

$$+(t+1)^k\frac{d^k}{dt^k}F_1(t).$$

Substituting the above representations in Eq. (13.11), we obtain

$$-g(-1)\,2^{\alpha}\cot\pi\beta\,(t+1)^{\beta}+g(1)\,2^{\beta}\cot\pi\alpha\,(1-t)^{\alpha}+$$

$$+F_0(t)-c_0g(-1)\,2^{\alpha}\frac{e^{-\pi i\beta}}{\sin\pi\beta}e^{i\beta\theta_1}(t+1)^{\beta}+F_1(t)+$$

$$+\sum_{k=1}^{n}c_k\Big[-g(-1)\,2^{\alpha}\frac{e^{-\pi i\beta}}{\sin\pi\beta}e^{i\beta\theta_1}\beta(\beta-1)\dots(\beta-k+1)(t+1)^{\beta}+$$

$$+(t+1)^k\frac{d^k}{dt^k}F_1(t)\Big]=f(t).$$

Since the left-hand side of the equation is bounded everywhere, and in particular in the neighbourhood of the point -1, we obtain equations for the determination of α and β:

$$\cot\pi\alpha=0,\quad \alpha=-\frac{1}{2},$$

$$\cos\pi\beta+e^{i\beta(\theta_1-\pi)}\Big[c_0+\sum_{k=1}^{n}c_k\beta(\beta-1)\dots(\beta-k+1)\Big]=0. \qquad (13.17)$$

Of course, if we retained terms with the coefficients b_j in Eq. (13.11), the resulting equations would be more unwieldy. Note that general

considerations on the solution of equations of this kind are given in the work of F. Erdogan [2].

It may also be mentioned that D. I. Sherman [26] obtained the exact solution for a specific equation belonging to the class under consideration. The revealed singularity of the solution coincided with that determined by Eqs. (13.17).

The foregoing approach does not exclude the possibility of solving Eq. (13.11) directly by the mechanical quadrature method. It should be noted that, after taking the density function outside the integral sign on each small section, the remaining expressions are integrated in closed form. Naturally, a finer discretization is needed in the neighbourhood of the ends.

The above procedure is fully applicable when the coefficients c_k and b_k are absent (i.e., when the equations are ordinary singular ones). We refer to the work of A. I. Kalandiya [4], which deals with the equation

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi(t)}{\tau-t}\,d\tau+\frac{1}{2\pi}\int_{-1}^{1}k(t,\ \tau)\,\varphi(\tau)\,d\tau=f(t) \qquad (13.18)$$

under all possible restrictions on the behaviour of the solution at the ends.

Chapter III

FUNDAMENTAL PRINCIPLES OF THE MATHEMATICAL THEORY OF ELASTICITY

14. Three-dimensional Problem

Let an elastic body occupy a region D bounded by a closed surface S in a three-dimensional space. If the region is finite, it will be denoted by D^+, and if it is infinite by D^-.

The solution of an elasticity problem consists in determining, at each point p (with Cartesian co-ordinates x_1, x_2, x_3), a vector $\boldsymbol{u}$ (with Cartesian co-ordinates u_1, u_2, u_3) characterizing a small displacement of this point during the deformation of a medium.

The vector field $\boldsymbol{u}$ (p) determines the so-called *small strain tensor* in the whole body:

$$\varepsilon_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right] \quad (i,\ j=1,\ 2,\ 3). \tag{14.1}$$

These strains in turn determine the components of the *stress tensor*, which in the case of an isotropic medium are respresentable in the form, according to Hooke's law,

$$\sigma_{ij}=\lambda_{ij}\theta+2\mu\varepsilon_{ij}, \quad \theta=\operatorname{div}\boldsymbol{u}, \tag{14.2}$$

where λ and μ are physical constants of the medium known as *Lamé's constants.**

The components of the stress tensor satisfy the differential *equations of equilibrium*

$$\frac{\partial\sigma_{ij}}{\partial x_j}=0 \quad (i=1,\ 2,\ 3). \tag{14.3}$$

It is assumed here that body forces are absent.

Substituting the displacement derivatives for the stresses in (14.3) in accordance with Hooke's law, we obtain the following equations

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* In the technical literature other constants are more often used, namely Young's modulus $E=\frac{(3\lambda+2\mu)\,\mu}{\lambda+\mu}$ and Poisson's ratio $\nu=\frac{\lambda}{2\,(\lambda+\mu)}$ which will be further assumed in the range $0\leqslant\nu<0.5$.

called *Lamé's equations*:

$$\Delta^* \boldsymbol{u} = \mu \Delta \boldsymbol{u} + (\lambda + \mu) \text{ grad div } \boldsymbol{u} = 0. \tag{14.4}$$

Since the strains (there are six of them) are the derivatives of three scalar functions, they are connected by the following six differential relations called *Saint-Venant's compatibility conditions*:

$$2 \frac{\partial^2 \varepsilon_{ij}}{\partial x_i \partial x_j} = \frac{\partial^2 \varepsilon_{ii}}{\partial x_j^2} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i^2} \quad (j \neq i), \tag{14.5}$$

$$\frac{\partial^2 \varepsilon_{ii}}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_i} \left(-\frac{\partial \varepsilon_{jk}}{\partial x_i} + \frac{\partial \varepsilon_{ji}}{\partial x_k} + \frac{\partial \varepsilon_{ik}}{\partial x_j} \right) \quad (i \neq j \neq k).$$

Let a plane be given at an arbitrary point p, by specifying it by a normal $\boldsymbol{n}$ (n_{x_1}, n_{x_2}, n_{x_3} are the direction cosines). A knowledge of the components of the stress tensor and the direction cosines enables us to obtain an expression for the projections of the stress vector $[\sigma_{in}$ $(i = 1, 2, 3)]$ at this point acting on the plane in question. We have

$$\sigma_{in} = \sigma_{ij} \cos(\boldsymbol{n}, x_j) \quad (i = 1, 2, 3) \tag{14.6}$$

Substitution of the components of the strain tensor in (14.6) according to (14.2) leads to a compact representation for the stress vector directly in terms of displacements:

$$\boldsymbol{T}_n \boldsymbol{u}(q) = 2\mu \frac{\partial \boldsymbol{u}}{\partial n} + \lambda \boldsymbol{n} \text{ div } \boldsymbol{u} + \mu (\boldsymbol{n} \times \text{rot } \boldsymbol{u}), \tag{14.7}$$

Expression (14.7) is symbolically written as a result of the action of an operator $\boldsymbol{T}_n$ (stress operator) on the displacement $\boldsymbol{u}$ ($\boldsymbol{T}_n \boldsymbol{u}$). The operator $\boldsymbol{T}_n$ may be regarded not only at the interior points of an elastic body; it may be defined on the bounding surface as the limit of values at a set of interior points approaching the corresponding boundary point. It is required that the directions of the normals at the interior points should coincide or at least should tend to the direction of the normal at the boundary point.*

The solution of an elasticity problem consists in determining the stress and displacement fields (or one of them, as the case may require) according to the foregoing equations and various boundary conditions prescribed on the surface.

An elasticity problem is called the *first fundamental problem* if limiting values of the displacement vector are prescribed on the surface. A problem is called the *second fundamental problem* if limiting values of the operator $\boldsymbol{T}_n$ are given on the surface. In all these cases the boundary conditions will be designated in a unified manner by $\boldsymbol{f}(q)$.

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* It is worth-while to speak only of the points of the surface S at which the tangent plane is uniquely defined.

There may be other formulations of boundary value problems. Thus, displacements may be given over a part of the surface S, and stresses over the remainder (*mixed problem*). In some cases the boundary conditions are prescribed as certain relations between displacements and stresses. For example, such is the case when values of the normal component of displacement and the tangential components of stress are known.

The solution of elasticity problems can be carried out directly in terms of displacements proceeding from Eqs. (14.4), taking into account the given boundary conditions, and determining the stress values at the final stage. If, however, the components of the stress tensor are first determined in the solution, to Eqs. (14.3) must be added the strain compatibility equations in terms of stresses (the so-called *Beltrami-Michell equations*) obtained from Eqs. (14.5) by replacing strains by stresses.

In conformity with the mathematical technique used in the book we assume that the *displacement components are continuous and continuously differentiable in a closed region* (including the surface S) *and their second derivatives are continuous only in an open region* (the so-called regular solution).

Below are given some general theorems which will be needed in what follows. We first introduce the concept of a generalized stress σ^* by defining its components as

$$\sigma^*_{jk} = \alpha \frac{\partial u_k}{\partial x_j} + \mu \frac{\partial u_i}{\partial x_k} \qquad (j \neq k),$$

$$\sigma^*_{jj} = (\lambda + \mu - \alpha) \operatorname{div} \boldsymbol{u} + (\alpha + \mu) \frac{\partial u_j}{\partial x_j}, \tag{14.8}$$

where α is an arbitrary constant.

The generalized stress tensor in turn gives rise to the generalized stress operator acting on a plane with normal $\boldsymbol{n}$:

$$\boldsymbol{P}_n \boldsymbol{u} = (\alpha + \mu) \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + (\lambda + \mu - \alpha)\, \boldsymbol{n} \operatorname{div} \boldsymbol{u} + \alpha\, (\boldsymbol{n} \times \operatorname{rot} \boldsymbol{u}). \tag{14.9}$$

It is obvious that when $\alpha = \mu$ the generalized stresses are identical with the true stresses ($\boldsymbol{P}_n \equiv \boldsymbol{T}_n$).

Let $\boldsymbol{u}\,(p)$ and $\boldsymbol{v}\,(p)$ be displacements given in a region D^+. We form the scalar product of the vectors $\boldsymbol{u}\,(p)$ and $\boldsymbol{P}_n \boldsymbol{v}\,(p)$:

$$\begin{gathered} \boldsymbol{u} \cdot \boldsymbol{P}_n \boldsymbol{v} = Q_1 \cos(\boldsymbol{n},\ x_1) + Q_2 \cos(\boldsymbol{n},\ x_2) + Q_3 \cos(\boldsymbol{n},\ x_3), \\ Q_i = \sigma^*_{ji} u_j \quad (i = 1,\ 2,\ 3). \end{gathered} \tag{14.10}$$

Let us consider Q_i as the components of a certain vector $\boldsymbol{Q}$ (Q_1, Q_2, Q_3) and calculate its divergence:

$$\operatorname{div} \boldsymbol{Q} = \frac{\partial \sigma^*_{jk}}{\partial x_k} u_j + E\,(\boldsymbol{u},\ \boldsymbol{v}). \tag{14.11}$$

The expression for $E(u, v)$ is a symmetrical bilinear form:

$$E(u, v) = (\lambda + 2\mu) \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k}\frac{\partial v_k}{\partial x_k} + \mu \sum_{i \neq k} \frac{\partial u_i}{\partial x_k}\frac{\partial v_i}{\partial x_k} + (\lambda + \mu - \alpha) \sum_{i \neq k} \frac{\partial u_i}{\partial x_i}\frac{\partial v_k}{\partial x_k} + \alpha \sum_{i \neq k} \frac{\partial u_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}.$$

By using (14.8), we ascertain by direct differentiation that the double sum in (14.11) is the scalar product $u \cdot \Delta^* v$.

We thus obtain

$$\operatorname{div} Q = u \cdot \Delta^* v + E(u, v). \tag{14.12}$$

By integrating identity (14.12) over the volume, and applying the Gauss-Ostrogradsky formula, with (14.10), we find:

$$\int_{D^+} u \cdot \Delta^* v \, d\Omega = \int_S u P_n v \, dS - \int_{D^+} E(u, v) \, d\Omega, \tag{14.13}$$

this is the analogue of Green's formula and is called the *first generalized Betti formula* in the theory of elasticity.

By setting $u = v$ in formula (14.13), we arrive at the *second generalized Betti formula*:

$$\int_{D^+} u \cdot \Delta^* u \, d\Omega = \int_S u \cdot P_n u \, dS - \int_{D^+} D(u, u) \, d\Omega. \tag{14.14}$$

By interchanging the displacements u and v, and taking into account the symmetry of the bilinear form $E(u, v)$, we arrive at the *third generalized Betti formula*:

$$\int_{D^+} \{u \cdot \Delta^* v - v \cdot \Delta^* u\} \, d\Omega = \int_S \{u \cdot P_n v - v \cdot P_n u\} \, dS. \tag{14.15}$$

It is obvious that, by setting $\alpha = \mu$, we obtain the conventional Betti formulas (see I. N. Sneddon, D. S. Berry [1]).

Note that in this case the form $E(u, u)$ becomes

$$E(u, u) = 2\mu \left[2 \left\{ \left(\frac{\partial u_1}{\partial x_2} \right)^2 + \left(\frac{\partial u_2}{\partial x_3} \right)^2 + \left(\frac{\partial u_3}{\partial x_1} \right)^2 \right\} + \sum_{k=1}^{3} \left(\frac{\partial u_k}{\partial x_k} \right)^2 \right] + \lambda (\operatorname{div} u)^2, \tag{14.16}$$

which shows its positive definiteness.

Of some interest is the case when $\alpha = t \frac{\mu(\alpha + \mu)}{\lambda + 3\mu}$. The pseudostress operator will be designated as N_n. The bilinear form here is also

positive definite:

$$2\frac{\mu(\lambda+\mu)}{\lambda+3\mu}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^3+\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_k}\right)^2\right]+$$
$$+\frac{2\mu}{\lambda+3\mu}\sum_{k=1}^{3}(\operatorname{grad} u_k)^2+\frac{(\lambda+\mu)(\lambda+2\mu)}{\lambda+3\mu}(\operatorname{div} \boldsymbol{u})^2. \quad (14.17)$$

It is obvious that Betti's formulas are also valid for a region bounded by several surfaces. If for a region D^- we require that the displacements should decrease at infinity as $1/R$ and the strains as $1/R^2$, the above formulas are extended to this case (with the corresponding change of sign in front of the stress operator). As in the classical potential theory, the proof is based on the consideration of a region bounded from the inside by the surface S, and from the outside by a surface of sufficiently large size (see N. M. Günter [1]); by analysing the corresponding terms over this auxiliary surface as it increases indefinitely, it is shown that they tend to zero.

The uniqueness theorems follow immediately from the positive definiteness of the quadratic forms (14.16) and (14.17) established above since by virtue of the linearity of the equations the question reduces to that of the existence of non-trivial solutions in the case of homogeneous boundary conditions.

Consider the first fundamental problem for a region D^+ with zero boundary conditions. Let $\boldsymbol{u}_0(p)$ be a non-trivial solution. Two integrals in formula (4.4) then vanish (by condition). Consequently, the third integral must also be zero, which proves that the integrand, i.e., the form $E(\boldsymbol{u}_0, \boldsymbol{u}_0)$ is identically zero. The vector determining a rigid-body displacement and only this vector makes the form $E(\boldsymbol{u}_0, \boldsymbol{u}_0)$ zero. Since the displacements must, in addition, vanish on the surface, they are identically zero in the entire region.

The foregoing also proves the uniqueness theorem for the second fundamental problem. In this case the displacements need not vanish, but must correspond to a rigid displacement of the body:

$$u_1 = a_1 + qx_3 - rx_2, \quad u_2 = a_2 + rx_1 - px_3, \quad u_3 = a_3 + px_2 - qx_1, \quad (14.18)$$

where a_1, a_2, a_3, p, q, r are arbitrary constants. The corresponding stresses are zero.

For an infinite region, both problems have only a zero solution (even in terms of displacements) since displacements (14.18) do not satisfy the restriction on the behaviour of displacement at infinity adopted in deriving the above formulas.

The question (having only a mathematical significance) of the uniqueness of the solution of a problem when the operator $\boldsymbol{N}_n$ van-

ishes on the surface is considered in a similar way; this question will arise at a later stage. From the analysis of the positive form (14.17) it follows that the non-trivial solution of an interior problem is of the form

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \tag{14.19}$$

where a_1, a_2, a_3 are arbitrary constants.

In the case of an exterior problem the non-trivial solution is non-existent.

Consider a space filled with an elastic medium with Lamé's constants λ and μ. Let a unit concentrated force be applied at a point $p(y_1, y_2, y_3)$ in the direction of the axis *1*. According to the Kelvin-Somigliana formula (see, for example, A. I. Lur'e [1]) the displacements at any point $p_1(x_1, x_2, x_3)$ other than p are expressed by

$$u_i^{(1)} = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left[\frac{(x_i-y_i)(x_1-y_1)}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{\delta_{i1}}{r}\right] \quad (i=1, 2, 3), \tag{14.20}$$

where $r = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$, or alternatively

$$u_1^1 = \frac{1}{8\pi\mu(\lambda+2\mu)}\left[(\lambda+\mu)\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1} + (\lambda+3\mu)\delta_{i1}\right]\frac{1}{r}. \tag{14.20'}$$

The expressions for the displacements u_i^2 and u_i^3 when the force is directed along the axis *2* and the axis *3*, respectively, can be obtained by cyclic permutation.

We now turn to the general case. Let a force $\varphi(\varphi_1, \varphi_2, \varphi_3)$ be applied at the point p. The displacements at the point p_1 can be represented as the product of a certain matrix $\mathbf{\Gamma}(p_1, p)$ called the Kelvin-Somigliana matrix (with the element u_i^j given above) and the vector $\varphi(p)$.

We thus have

$$\boldsymbol{u}(p_1) = \mathbf{\Gamma}(p_1, p)\,\boldsymbol{\varphi}(p) \tag{14.21}$$

or in expanded form

$$\begin{aligned} u_1 &= \Gamma_{11}\varphi_1 + \Gamma_{12}\varphi_2 + \Gamma_{13}\varphi_3, \\ u_2 &= \Gamma_{21}\varphi_1 + \Gamma_{22}\varphi_2 + \Gamma_{23}\varphi_3, \\ u_3 &= \Gamma_{31}\varphi_1 + \Gamma_{32}\varphi_2 + \Gamma_{33}\varphi_3. \end{aligned}$$

The name Kelvin-Somigliana matrix (retaining the same notation) will be further used for a matrix obtained from the original one by multiplying it by 2 for convenience in writing integral equations derived from it.

We next pass a plane with normal $\boldsymbol{n}(n_{x_1}, n_{x_2}, n_{x_3})$ through the point p_1 and determine the stress vector acting on this plane at the point p_1. It follows from (14.21) that the required expression is given by the product of a matrix $\mathbf{\Gamma}_1(p_1, p) = T_{n(p_1)}\mathbf{\Gamma}(p_1, p)$ and the

vector $\varphi(p)$. By using (14.7), after lengthy computations, we arrive at an expression for the elements of the matrix $\Gamma_1(p_1, p)$:

$$\left\| \begin{matrix} m+n\left(\frac{\partial r}{\partial x_1}\right)^2 & n\frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_1} & n\frac{\partial r}{\partial x_3}\frac{\partial r}{\partial x_1} \\ n\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & m+n\left(\frac{\partial r}{\partial x_2}\right)^2 & n\frac{\partial r}{\partial x_3}\frac{\partial r}{\partial x_2} \\ n\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_3} & n\frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_3} & m+n\left(\frac{\partial r}{\partial x_3}\right)^2 \end{matrix} \right\| \times$$

$$\times \frac{d}{dn(p_1)}\frac{1}{r(p_1, p)} +$$

$$+m \left\| \begin{matrix} 0 & \omega_{12}(p_1, p) & \omega_{13}(p_1, p) \\ -\omega_{12}(p_1, p) & 0 & \omega_{23}(p_1, p) \\ -\omega_{13}(p_1, p) & -\omega_{23}(p_1, p) & 0 \end{matrix} \right\|, \quad (14.22)$$

where

$$m = \frac{1}{2\pi}\frac{\mu}{\lambda+2\mu}, \quad n = \frac{3}{2\pi}\frac{\lambda+\mu}{\lambda+2\mu}, \quad \omega_{ij} = \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) \times$$

$$\times n_j(p_1) - \frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) n_i(p_1).$$

We now draw on Betti's third formula (14.15) assuming $\alpha = \mu$ for definiteness. Let p be a point situated in a region D^+ bounded by a surface S. Construct a sphere σ_ε of sufficiently small radius ε centred at the point p. Consider a region D_ε^+ contained between the surfaces S and σ_ε. We apply Betti's formula to a displacement $\boldsymbol{u}(p_1)$ satisfying Lamé's equation in the region D^+ and to a displacement $\boldsymbol{v}(p_1)$ produced by a force $\boldsymbol{a}_i$ (i.e., by a vector whose ith component is 1 and the others are zero) applied at the point p. Naturally,

$$\boldsymbol{v}(p_1) = \boldsymbol{\Gamma}(p_1, p)\,\boldsymbol{a}_i.$$

We first assume $i = 1$ and then consider the cases $i = 2$ and $i = 3$ in a similar way.

It is clear at once that the volume integrals in Betti's formula vanish. Since the displacement $\boldsymbol{v}(p_1)$ has a pole of the first order in the neighbourhood of the point p and the stresses corresponding to the displacement $\boldsymbol{u}(p_1)$ are bounded, we may state that the integral

$$\int_{\sigma_\varepsilon} \boldsymbol{\Gamma}(p_1, p)\,\boldsymbol{a}_1 \boldsymbol{T}_{n(p_1)}\boldsymbol{u}(p_1)\, dS_{p_1}$$

tends to zero as $\varepsilon \to 0$, and in consequence we exclude it from further consideration. Since the displacement $\boldsymbol{u}(p_1)$ is continuous in the

neighbourhood of the point p, the evaluation of the integral

$$\int_{\sigma_\varepsilon} \{T_{n(p_1)}\Gamma(p_1, p)\,a_1\}\,u(p_1)\,dS_{p_1} \tag{14.23}$$

reduces to the evaluation of the integral

$$\int_{\sigma_\varepsilon} \{T_{n(p_1)}\Gamma(p_1, p)\,a_1\}\,dS_{p_1} = \int_{\sigma_\varepsilon} \Gamma_1(p_1, p)\,a_1\,dS_{p_1}. \tag{14.24}$$

Since the surface σ_ε is a sphere, a spherical co-ordinate system is used to evaluate the integral of each component of the matrix $\Gamma_1(p_1, p)$ in (14.22). The following equalities are easily obtained:

$$\begin{gathered}
\int_{\sigma_\varepsilon} \frac{d}{dn}\frac{1}{r(p_1, p)}\,dS_{p_1} = -4\pi, \\
\int_{\sigma_\varepsilon} \left(\frac{\partial r}{\partial x_i}\right)\left(\frac{\partial r}{\partial x_j}\right)\frac{d}{dn}\frac{1}{r(p_1, p)}\,dS_{p_1} = 0 \quad (i \neq j), \\
\int_{\sigma_\varepsilon} \left(\frac{\partial r}{\partial x_i}\right)^2 \frac{d}{dn}\frac{1}{r(p_1\, p)}\,dS_{p_1} = -\frac{4}{3}\pi, \quad \int_{\sigma_\varepsilon} \omega_{ij}\,dS_{p_1} = 0.
\end{gathered} \tag{14.25}$$

The integrals of the functions ω_{ij} are zero because they are odd. Thus, from Betti's third formula [with (14.25)] it follows that

$$\begin{aligned}
2u_1(p) = &-\int_S T_{n(p_1)}\Gamma(p_1, p)\;a_1 u(p_1)\,dS_{p_1} + \\
&+ \int_S \Gamma(p_1, p)\cdot a_1 T_{n(p_1)} u(p_1)\,dS_{p_1}
\end{aligned} \tag{14.26}$$

By applying the same reasoning to the vectors a_2 and a_3, we obtain similar representations. They may be written in a unified analytic form (with the obvious change of the arguments):

$$2u(p) = -\int_S \Gamma_2^{\mathrm{I}}(p, q)\,u(q)\,dS_q + \int_S \Gamma(p, q)\,T_n u(q)\,dS_q, \tag{14.27}$$

where the matrix $\Gamma_2^{\mathrm{I}}(p, q)$ is the conjugate to the matrix $\Gamma_1(p, q)$ (i.e., a matrix obtained from it by interchanging the arguments and by transposition). In view of its importance in the following discussion, we give an expanded expression for the elements of the

matrix $\mathbf{\Gamma}_2^{\mathrm{I}}(p, q)$:

$$\mathbf{\Gamma}^{\mathrm{I}}_{2(k,j)}(p, q) = \left[m\delta_{k,j} + 3n \frac{(y_k - x_k)(y_j - x_j)}{r^2} \right] \frac{\sum_{l=1}^{3} (x_l - y_l)\, n_l(q)}{r^3} + \\ + m \left[n_k(q) \frac{(x_j - y_j)}{r^3} - n_j(q) \frac{(x_k - y_k)}{r^3} \right]. \quad (14.28)$$

Let now the point p be chosen in a region D^-. It follows directly from Betti's third formula that

$$0 = -\int_S \mathbf{\Gamma}_2^{\mathrm{I}}(p, q)\, \boldsymbol{u}(q)\, dS_q + \int_S \mathbf{\Gamma}(p, q)\, T_n \boldsymbol{u}(q)\, dS_q. \quad (14.29)$$

In this case the vector function $\boldsymbol{u}(p)$ satisfies, as before, Lamé's equations in the region D^+.

Similar constructions are also possible in the case when the displacement $\boldsymbol{u}(p)$ is defined in the region D^- (with the restrictions on the behaviour at infinity already noted). The corresponding formulas are as follows:

$$2\boldsymbol{u}(p) = \int_S \mathbf{\Gamma}_2^{\mathrm{I}}(p, q)\, \boldsymbol{u}(q)\, dS_q - \int_S \mathbf{\Gamma}(p, q)\, T_n \boldsymbol{u}(q)\, dT_q \quad (p \in D^-), \quad (14.30)$$

$$0 = \int_S \mathbf{\Gamma}_2^{\mathrm{I}}(p, q)\, \boldsymbol{u}(q)\, dS_q - \int_S \mathbf{\Gamma}(p, q)\, T_n \boldsymbol{u}(q)\, dS_q \quad (p \in D^+). \quad (14.30')$$

We now turn to the N-operator. By repeating the foregoing transformations, we obtain formulas similar to (14.27), (14.29), and (14.30), (14.30'):

$$2\boldsymbol{u}(p) = -\int_S \mathbf{\Gamma}_2^{\mathrm{II}}(p, q)\, \boldsymbol{u}(q)\, dS_q + \\ + \int_S \mathbf{\Gamma}(p, q)\, N_n \boldsymbol{u}(q)\, dS_q \quad (p \in D^+), \quad (14.31)$$

$$0 = -\int_S \mathbf{\Gamma}_2^{\mathrm{II}}(p, q)\, \boldsymbol{u}(q)\, dS_q + \int_S \mathbf{\Gamma}(p, q)\, N_n \boldsymbol{u}(q)\, dS_q \quad (p \in D^-),$$

$$2\boldsymbol{u}(p) = \int_S \mathbf{\Gamma}_2^{\mathrm{II}}(p, q)\, \boldsymbol{u}(q)\, dS_q - \\ - \int_S \mathbf{\Gamma}(p, q)\, N_n \boldsymbol{u}(q)\, dS_q \quad (p \in D^+), \quad (14.32)$$

$$0 = \int_S \mathbf{\Gamma}_2^{\mathrm{II}}(p, q)\, \boldsymbol{u}(q)\, dS_q - \int_S \mathbf{\Gamma}(p, q)\, N_n \boldsymbol{u}(q)\, dS_q \quad (p \in D^-).$$

The matrix $\mathbf{\Gamma}_2^{\mathrm{II}}(p, q)$ has been used above to denote the product $N_n(q)\,\mathbf{\Gamma}(p, q)$. We give the expression for this matrix:

$$\mathbf{\Gamma}_2^{\mathrm{II}}(p,q)=\left\|\begin{matrix} m_1+n_1\left(\frac{\partial r}{\partial x_1}\right)^2 & n_1\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & n_1\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_3} \\ n_1\frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_1} & m_1+n_1\left(\frac{\partial r}{\partial x_2}\right)^2 & n_1\frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_3} \\ n_1\frac{\partial r}{\partial x_3}\frac{\partial r}{\partial x_1} & n_1\frac{\partial r}{\partial x_3}\frac{\partial r}{\partial x_1} & m_1+n_1\left(\frac{\partial r}{\partial x_3}\right)^2 \end{matrix}\right\| \times$$

$$\times\frac{d}{d\boldsymbol{n}(p)}\frac{1}{r}, \qquad (14.33)$$

$$m_1=\frac{2}{\lambda+3\mu}, \qquad n_1=\frac{3(\lambda+\mu)}{\mu(\lambda+3\mu)}.$$

Note that matrix (14.33) involves no terms having poles of the second order. The matrix $\mathbf{\Gamma}_2^{\mathrm{II}}(p, q)$ (*matrix of the second kind*) as well as the matrix $\mathbf{\Gamma}_1^{\mathrm{II}}(p, q)$ (*matrix of the first kind*) will be used in what follows. The specific features of the notation for these matrices will become clear below (see Chap. VI). Note that similar formulas occur for any value of α in the operator $\boldsymbol{P_n}$.

We now construct the *matrix of the third kind*. Let there be a surface S possessing the property that the outward normal at any point does not intersect it any more. Let a point q lie on S and a point p be arbitrary. Consider the function

$$v(p, q) = r\cos(\boldsymbol{r}_0, \boldsymbol{n}_p)\ln[r + r\cos(\boldsymbol{r}_0, \boldsymbol{n}_p)] - r, \qquad (14.34)$$

where $\boldsymbol{n}_p$ is the unit inward normal at the point p, $\boldsymbol{r}_0$ is the unit vector of a segment drawn from p to q. It follows from (14.34) that the function $v(p, q)$ is independent of the arrangement of co-ordinate axes. For simplicity in writing, we choose a co-ordinate system so that the origin lies at the point p and the positive direction of the x_1 axis coincides with the inward normal. Relation (14.34) (in local co-ordinates) then becomes

$$v = x_1\ln(r + x_1) - r. \qquad (14.34')$$

It is obvious that the function $v(p, q)$ will always have meaning at the points inside the surface S because of the inequality $r + x_1 \neq 0$.

We form a matrix using this function:

$$\boldsymbol{Z}(p,q)=\left\|\begin{matrix} \frac{\partial^2 v}{\partial x_1^2} & -\frac{\partial^2 v}{\partial x_1\partial x_2} & -\frac{\partial^2 v}{\partial x_1\partial x_3} \\ \frac{\partial^2 v}{\partial x_1\partial x_2} & \frac{\partial^2 v}{\partial x_1^2}-\frac{\partial^2 v}{\partial x_3^2} & \frac{\partial^2 v}{\partial x_2\partial x_3} \\ \frac{\partial^2 v}{\partial x_1\partial x_3} & \frac{\partial^2 v}{\partial x_2\partial x_3} & \frac{\partial^2 v}{\partial x_1^2}-\frac{\partial^2 v}{\partial x_2^2} \end{matrix}\right\|. \qquad (14.35)$$

Direct calculations show that each column of this matrix regarded as a vector satisfies Lamé's equations (14.4).

We now turn to a new matrix

$$M(p, q) = \frac{1}{3(\lambda+\mu)}\left[\frac{1}{2} Z(p, q) - (\lambda + 2\mu)\,\Gamma(p, q)\right], \qquad (14.36)$$

introduced by H. Weyl [1]. The difference of the fundamental solution corresponding to matrix (14.36) from the preceding ones is that its construction requires at least a local specification of a surface. Moreover, this fundamental solution does not satisfy the conditions at infinity.

Let us now consider the formulation of an elasticity problem for a piecewise homogeneous medium. Suppose that in a finite or an infinite elastic body there are cavities filled with elastic bodies of the same size with different values of Lamé's coefficients. On the contact surfaces of the media various matching conditions may be specified. For example, there may be a jump in the displacement vector (sometimes non-existent) or in its normal component, while the stress vector undergoes no discontinuity. It may also be assumed that there is a clearance between the elastic bodies (over an unknown part of the surface). Over the remaining (adherent) part of the surface the stress vector is assumed, as before, to be continuous and the displacement vector may have a given discontinuity. One more condition is realized when stresses (generally equal to zero) are prescribed on the sides of a cavity formed in a body. The conditions permitting a determination of the cavity size are found from the requirement that the sign of the normal stress over the remaining part of the surface should everywhere be negative (to produce compression). It is not essential that the interface between the media should necessarily be situated strictly inside the overall region. It may be assumed that one or several such interfaces terminate at the external boundaries. And again various conditions may be fulfilled at the interface.

15. Plane Problem

Consider the case when all stress and displacement components are functions of only two co-ordinates, $x_1 = x$ and $x_2 = y$, and the displacement $u_3 = w \equiv 0$. From the equalities (14.1) and (14.2) it immediately follows that the strains $\varepsilon_{13} = \gamma_{xz}$, $\varepsilon_{23} = \gamma_{yz}$, $\varepsilon_{33} = \varepsilon_z$ and the stresses $\sigma_{13} = \tau_{xz}$, $\sigma_{23} = \tau_{yz}$ are zero. The third equilibrium equation of (14.3) is automatically satisfied, and the first two become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \qquad (15.1)$$

From Hooke's law, if $\varepsilon_z = 0$, it follows that

$$\sigma_z = \lambda\theta = \frac{\lambda}{2(\lambda+\mu)}(\sigma_x + \sigma_y).$$

The stress-strain relations take the form

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}, \quad \sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}, \tag{15.2}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right).$$

All of the strain compatibility equations (14.5) are automatically fulfilled. The third equation reduces to

$$\Delta(\sigma_x + \sigma_y) = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{15.3}$$

Consider a plane with a normal n lying in the xy plane. The stresses acting on this plane are

$$\begin{aligned} \sigma_{xn} &= \sigma_x \cos(n, x) + \tau_{xy}\cos(n, y), \\ \sigma_{yn} &= \tau_{xy}\cos(n, x) + \sigma_y \cos(n, y), \quad \sigma_{zn} \equiv 0. \end{aligned} \tag{15.4}$$

This state is realized in cylindrical bodies extending indefinitely along the z axis if the external stresses σ_{xn} and σ_{yn} or the displacements u and v on the surface are constant along the generators. In the case of a finite extension along the z axis it is necessary that the displacement w and the shearing stresses τ_{xz} and τ_{yz} should be zero at the ends.

It is natural to carry out the solution of problems of this kind only at any one cross section. Let the region occupied by the section be denoted by D, and the contour bounding it by L. The state described above is called *plane strain*.

Consider, now, another state, called *plane stress*. Let there be a cylinder of small thickness. The co-ordinate axes are chosen so that the xy plane coincides with the middle plane of the cylinder, further referred to as a plate. Assume that the ends are free from stresses ($\sigma_z = \tau_{xz} = \tau_{yz} \equiv 0$), and the resultant of the stresses along the generator lies in the xy plane. In accordance with Saint-Venant's principle we assume that the stresses and displacements far from the edge behave so as if the stresses σ_{xn} and σ_{yn} were uniformly distributed along the height, and the stresses $\sigma_{zn} = 0$.

The foregoing shows that the stresses σ_x, σ_y, τ_{xy} and the displacements u, v may approximately be assumed to be functions only of the co-ordinates x and y, and the stresses σ_z, τ_{xz}, τ_{yz} equal to zero. It is obvious that the equilibrium equations are identical with

Eqs. (15.1), and the stress-displacement relations become

$$\sigma_x = \lambda^* \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x}, \quad \sigma_y = \lambda^* \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y}, \tag{15.5}$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}.$$

In consequence, the strain compatibility equation coincides with Eq. (15.3).

Thus, plane strain and plane stress are described by the same differential equations, differing only by the stress-strain relations. Their consideration will therefore further be carried out simultaneously (without proper specification).

It appears from the above discussion that the solution of the plane problem reduces to the solution of the system of equations (15.1) and (15.3). We introduce *Airy's stress function* $U(x, y)$ in terms of which the stress components are expressed as

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x\, \partial y}. \tag{15.6}$$

By direct substitution we ascertain that the first two equations of the system become identities, and the third equation reduces to a biharmonic equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U(x, y) = \Delta^2 U = 0. \tag{15.7}$$

Thus, the plane problem of the theory of elasticity reduces to the solution of Eq. (15.7). An arbitrary biharmonic function in a certain region may be expressed in terms of two analytic functions in the same region according to Goursat's formula (see N. I. Muskhelishvili [3]):

$$U(x, y) = \operatorname{Re} [\bar{z}\varphi(z) + \chi(z)] \quad (z = x + iy), \tag{15.8}$$

where $\varphi(z)$, $\varkappa(z)$ are analytic functions in D.

The components of the stress tensor are expressed in terms of the functions $\varphi(z)$ and $\psi(z)$ ($\psi(z) = \chi'(z)$) as follows:

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \varphi'(z) = 4 \operatorname{Re} [\Phi(z)],$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2 [\bar{z}\varphi''(z) + \psi'(z)] = 2 [\bar{z}\Phi'(z) + \Psi(z)], \tag{15.9}$$

$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z).$$

These are the well-known *Kolosov-Muskhelishvili formulas.*

The displacement components are expressed as

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{15.10}$$

Here

$$\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu, \quad \varkappa = \frac{\lambda^* + 3\mu}{\lambda^* + \mu} = \frac{3 - \nu}{1 + \nu}$$

in plane strain and plane stress, respectively.

Direct expression of the stress boundary conditions following from (15.4) in terms of the functions $\varphi(z)$ and $\psi(z)$ presents some difficulty. These conditions may be written more compactly in an alternate form. Consider an arc L' with ends a and b in a region occupied by an elastic body. Determine the resultant vector $X + iY$ of the forces applied to the arc L' on the positive side of the normal:

$$X+iY=\int_{L'}(X_n+iY_n)\,dS=-i\,[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}]_b^a. \tag{15.11}$$

Suppose that the arc L' belongs to the boundary of the region, the contour L. We fix the point a and assume the point b variable. We arrive at the following representation of the boundary condition:

$$\varphi(z)+(z)\,\overline{\varphi'(z)}+\overline{\psi(z)}=i\int_a^z(X_n+iY_n)\,dS+\text{constant}. \tag{15.12}$$

It is natural to assume that the displacements and stresses determined by the solution must be single-valued functions. In the case of a simply connected (finite) region these restrictions are equivalent to the single-valuedness of the functions $\varphi(z)$ and $\psi(z)$. In other cases (a finite or an infinite plane with m holes) it must be assumed that these functions may have multiple-valued terms:

$$\begin{aligned}\varphi(z)&=-\frac{1}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k+iY_k)\ln(z-z_k)+\varphi^*(z),\\ \psi(z)&=\frac{\varkappa}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k-iY_k)\ln(z-z_k)+\psi^*(z).\end{aligned} \tag{15.13}$$

Here z_k are points arbitrarily situated inside each of the boundary (inner) contours L_k; X_k and Y_k are the components of the resultant vector of the forces applied to the contour L_k. The functions $\varphi^*(z)$ and $\psi^*(z)$ are single-valued analytic functions.

Let us introduce a mathematical restriction. We shall investigate only so-called regular solutions when the functions $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ are continuously extendible to boundary points.

We now turn to the formulation of boundary value problems of the theory of analytic functions corresponding to the first and second fundamental problems (in the terminology introduced in Sec. 14). We first consider the case when the region D is a simply connected (finite) region bounded by a smooth closed contour L.

Suppose that stresses X_n and Y_n are prescribed on the contour L (second fundamental problem). We draw on condition (15.12) assuming that the points z lie on the contour L. Points of the complex plane situated on the boundary contours will be further denoted by t.

Since the right-hand side of formula (15.12) can be calculated in some way or other, the solution of the elasticity problem reduces to the determination of analytic functions $\varphi(z)$ and $\psi(z)$ satisfying the limiting relation

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=i\int_{t_0}^{t}(X_n+iY_n)\,dS+\text{constant}=$$
$$=f(t)+\text{constant}. \qquad (15.14)$$

The value of the constant is of no importance since in determining the stresses by formula (15.9) the functions $\varphi(z)$ and $\psi(z)$ must be differentiated, and in determining the displacements according to (15.10) the difference in the choice of the constant affects only a rigid-body displacement.

The boundary value problem for the analytic functions $\varphi(z)$ and $\psi(z)$ when displacements are prescribed on the contour (first fundamental problem) is obtained in a similar way:

$$\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}=\frac{1}{2\mu}[q_1(t)+iq_2(t)]=f(t), \qquad (15.15)$$

where $q_1(t)$ and $q_2(t)$ are given functions.

In solving problems in the case of multiply connected regions, we must pass to single-valued analytic functions according to (15.13) to make the modified boundary condition single-valued. In the case of the second fundamental problem the constants X_k and Y_k are known from the boundary conditions. The constants that enter into the boundary conditions (15.14) now extended to all contours L_k cannot be prescribed in an arbitrary manner (with one exception) and are determined in solving the problem.

Just as the formulation of problems of three-dimensional elasticity for piecewise homogeneous bodies was given in the concluding part of Sec. 14, we shall now discuss this question in the case of plane strain. For ease of presentation, we consider the case of a region D_1 bounded externally by a contour L_1 and internally by a contour L_0 and filled with an elastic medium with parameters $\varkappa_1$ and μ_1. Inside the contour L_0 (region D_0) there is an elastic medium with parameters $\varkappa_0$ and μ_0. The state of stress in either region is representable by a pair of functions, $\varphi_1(z)$, $\psi_1(z)$ or $\varphi_0(z)$, $\psi_0(z)$. There are several possible conditions on the contour L_1. For example, in the case of cohesion the conditions are of the form

$$\varphi_0(t)+t\overline{\varphi_0'(t)}+\overline{\psi_0(t)}=\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)},$$
$$\frac{1}{\mu_0}[\varkappa_0\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}]=$$
$$=\frac{1}{\mu_1}[\varkappa_1\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}]+\frac{f(t)}{\mu_1}, \qquad (15.16)$$

the function $f(t)$, which determines an allowable jump in the displacement (negative allowance), may be zero.

The extension of the problem formulation to the general case of several inclusions is obvious.

Let us now consider the question of mixed (contact) plane problems. Suppose that the contour L bounding a body is divided into several portions L_j so that either condition (15.14) or (15.15) is fulfilled in each of the adjacent portions. Physically, this kind of boundary condition corresponds to the problem for the case when external stresses are prescribed in some portions and rigid punches are applied (with cohesion) in the others. In general, condition (15.15) is usually prescribed to within some constants (to be determined in the course of solution), and the given quantities are assumed to be the resultant vector and the resultant moment of the forces. In the absence of cohesion (for example, when the shearing stress is zero), the corresponding conditions become more complicated.

Another possible formulation of mixed (contact) problems is one where the division of the contour L into portions is not known beforehand, but has to be determined as the solution proceeds. To determine this (in constructing the solution), some restrictions are introduced; for example, when there is no friction, the contact pressure must be negative.

The foregoing considerations also hold in plane stress. Moreover, in this case the actual non-homogeneity may occur due to a change in thickness, even though all constants of the medium remain unchanged, and this leads to the boundary condition

$$\begin{aligned} \alpha\,[\varphi_0(t)+t\overline{\varphi_0'(t)}+\overline{\psi_0(t)}] &= \varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}, \\ \varkappa_0\varphi_0-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)} &= \varkappa_0\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}, \end{aligned} \tag{15.17}$$

where α is the ratio of the thicknesses h_0/h_1. Naturally, this formulation takes no account of the three-dimensional effect of stress concentration on the line of contact.

In Sec. 14 we have proved the uniqueness theorems for the fundamental three-dimensional problems of the theory of elasticity. In reference to the plane problem these theorems are stated as follows.

In the case of the first fundamental problem the functions $\varphi(z)$ and $\psi(z)$ are uniquely determined to within the complex constants γ and γ' related by the equality $\varkappa\gamma - \overline{\gamma}' = 0$.

In the case of the second fundamental problem the functions $\varphi(z)$ and $\psi(z)$ are determined to within the terms $Ciz+\gamma$ and γ', respectively (where C is a real constant). The displacements differ by a rigid-body displacement. Of course, if the region is infinite and if from some considerations a condition is introduced that the displacements are zero at infinity, all terms vanish.

Note that in the case of the mixed problem the functions $\varphi(z)$ and $\psi(z)$ are determined to the same accuracy as for the first fundamental problem.

16. Bending of Thin Plates

Consider an elastic body in the form of a cylinder of small thickness h. As before, we choose a Cartesian co-ordinate system x, y, z so that the x and y axes lie in the middle plane.

We study a special case of the deformation of the body in question. Suppose that the hypothesis of plane sections is fulfilled (see A. E. H. Love [1]). Consider an element of a section of the plate parallel to

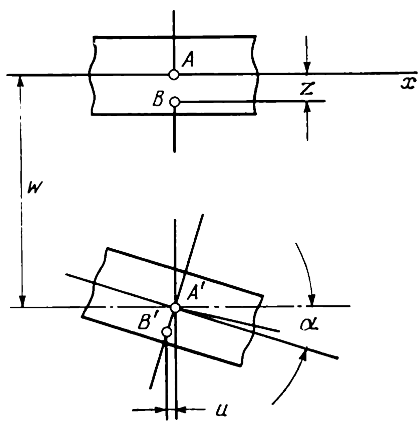

Fig. 5. Displacements in a plate

the xz plane (Fig. 5). Take points A and B situated on the same normal to the undeformed middle plane, the point A lying in the middle plane, and the point B at a distance z from it. We write expressions for the displacements of the point B in the x and y directions:

$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y}. \tag{16.1}$$

The displacements of points of the middle plane in the x and y directions are excluded. The expressions for the strains (14.1) then become

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 v}{\partial y^2}, \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}. \tag{16.2}$$

Assuming σ_z, τ_{xz}, and τ_{yz} to be small, we arrive at the representation of stress components through the derivatives of the displacement w only:

$$\begin{aligned}\sigma_x &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\\ \sigma_y &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\\ \tau_{xy} &= -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\,\partial y}.\end{aligned} \tag{16.3}$$

Using these relations, we can determine the bending and twisting moments per unit length of a section parallel to the xz or yz plane:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = -\frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) z\, dz. \quad (16.4)$$

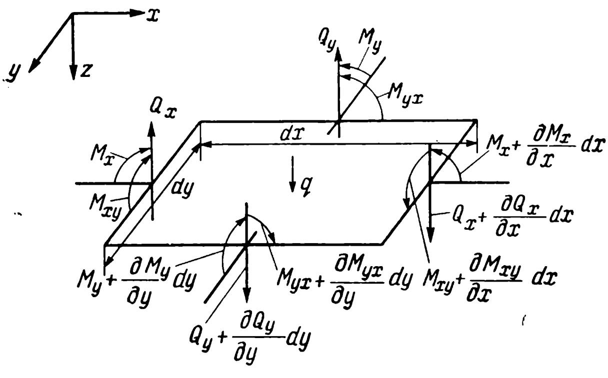

Fig. 6. Loading on an element of a plate

Since w is the deflection of the middle plane (and hence is independent of z), we rearrange expression (16.4)

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right),$$

where the constant $D = Eh^3/12\,(1 - \nu^2)$ is called the *flexural rigidity* of a plate. In a similar way we obtain

$$\begin{aligned} M_y &= -\int_{-h/2}^{h/2} \sigma_y z\, dz = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \\ M_{xy} &= -\int_{-h/2}^{h/2} \tau_{xy} z\, dz = D (1 - \nu) \frac{\partial^2 w}{\partial x\, \partial y}. \end{aligned} \quad (16.6)$$

Note that the faces of an element (Fig. 6) are acted on by shearing forces Q_x and Q_y determined by the stresses τ_{zx} and τ_{zy}:

$$Q_x = \int_{-h/2}^{h/2} \tau_{zx}\, dz, \quad Q_y = \int_{-h/2}^{h/2} \tau_{zy}\, dz. \quad (16.7)$$

From the condition that the sum of the forces in the normal direction is zero, we obtain

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \quad (16.8)$$

where q is the transverse load. At the same time, from the condition that the sums of the moments about the x and y axes are zero it follows that

$$\frac{\partial M_x}{\partial x}-\frac{\partial M_{yx}}{\partial y}-Q_x=0, \quad \frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0. \tag{16.9}$$

By Eqs. (16.9), the shearing forces are represented in terms of the displacement $w(x, y)$ as

$$Q_x=-D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right), \quad Q_y=-D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right). \tag{16.10}$$

Substituting these representations in Eq. (16.8), we obtain a differential equation for the displacement w, which is the fundamental equation in the theory of plate bending known as the *equation of Sophie Germain*:

$$\Delta^2 w=q/D. \tag{16.11}$$

Thus, in the absence of a shearing force ($q=0$) the solution of a bending problem reduces to a biharmonic equation. It is obvious that in the general case, too, if a particular solution of the non-homogeneous equation is known, we arrive at the solution of the homogeneous problem. Let the corresponding particular solution be denoted by $w^1(x, y)$, and the general solution of the biharmonic problem by $w_0(x, y)$. In many cases a particular solution is found in an elementary way. In general, the construction of a particular solution presents no difficulty in principle (see N. I. Muskhelishvili [3]).

According to Goursat's formula (15.8), we represent the function $w_0(x, y)$ by means of two analytic functions, $\varphi(z)$ and $\chi(z)$, in the region D occupied by the middle plane of the plate:

$$w_0(x, y)=2\mathrm{Re}\,[\bar{z}\varphi(z)+\chi(z)]. \tag{16.12}$$

By using the preceding formulas, we obtain representations for the bending moments M_x, M_y, the twisting moment M_{xy}, the shearing forces Q_x, Q_y, and the displacements u, v in terms of these functions, $\varphi(z)$ and $\psi(z)=\chi'(z)$. It is useful for later work to represent the formulas for the force factors as

$$\begin{gathered}
M_y-M_x+2iM_{xy}=4D(1-\nu)[\bar{z}\varphi''(z)+\psi'(z)]+\\
+(M_y^1-M_x^1-2iM_{xy}^1),\\
M_x+M_y=-4D(1+\nu)[\varphi'(z)+\overline{\varphi'(z)}]+(M_x^1+M_y^1),\\
Q_x-iQ_y=-8D\varphi''(z)+(Q_x^1-iQ_y^1).
\end{gathered} \tag{16.13}$$

The displacement components are represented as the following combination (Z denotes the co-ordinate along an axis normal to the

middle plane):

$$u+iv=-2[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}]Z-\left(\frac{\partial w^1}{\partial x}+i\frac{\partial w^1}{\partial y}\right)Z. \quad (16.14)$$

Formulas (16.13) and (16.14) involve terms with the superscript 1 showing that they correspond to the particular solution $w^1(x, y)$.

We now turn to the discussion of boundary conditions. As before, the boundary of the region is assumed to be smooth and is denoted by L.

(1) Suppose that the edge of the plate is free from geometric constraints and is acted on by a bending moment $m(s)$ and a shearing force $p(s)$ (the position of points of the contour is measured from some starting point along its length). Let M_n, $M_{n\tau}$, and Q_n be, respectively, the bending and twisting moments and the shearing force at a section with normal n. We have the equalities

$$M_n=m(s), \quad N_n=Q_n+\frac{\partial M_{n\tau}}{\partial s}=p(s). \quad (16.15)$$

The force factors M_n, $M_{n\tau}$, and Q_n are expressed in terms of M_x, M_y, M_{xy}, Q_x, and Q_y by formulas similar to the formulas for the transformation of stress components when the co-ordinate axes are rotated:

$$\begin{aligned} M_n&=M_x\cos^2(n, x)+M_y\cos^2(n, y)+2M_{xy}\cos(n, x)\cos(n, y),\\ M_{n\tau}&=(M_y-M_x)\cos(n, x)\cos(n, y)+\\ &\quad+M_{xy}[\cos^2(n, x)-\cos^2(n, y)],\\ Q_n&=\pm[Q_x\cos(n, x)+Q_y\cos(n, y)] \end{aligned} \quad (16.16)$$

Here the plus sign refers to a finite region D, and the minus sign to an infinite region, i.e., a plate with a hole.

By integrating the second of conditions (16.15) with respect to the arc length, we obtain (c is a constant)

$$\begin{aligned} P+M_{n\tau}&=f(s)+c,\\ P=\int_0^s Q_n\,ds, &\quad f(s)\int_0^s p(s)\,ds. \end{aligned} \quad (16.17)$$

With the help of (16.13) we transform to the functions $\varphi(z)$ and $\psi(z)$ and substitute (16.16) in the first of conditions (16.15) and in (16.17). The result is the required representation for the boundary value problem (c_0 is a real constant):

$$\begin{aligned} &-\varkappa\overline{\varphi(z)}+\bar{z}\varphi'(z)+\psi(z)=\\ &=\frac{1}{2D(1-\nu)}\left[\int_0^s (m^1+if^1)(dz)-\int_0^s (m+if)(dz)\right]-ic_0 z, \end{aligned} \quad (16.18)$$

where m^1 and f^1 correspond to a particular solution.

(2) Suppose that a value of the deflection $w_*(s)$ and a value of its normal derivative are given at the edge of the plate. We form a complex function (α is the angle between the outward normal and the x axis):

$$\frac{\partial w_*}{\partial n}+i\frac{\partial w_*}{\partial s}=e^{i\alpha}\left(\frac{\partial w_*}{\partial x}-i\frac{\partial w_*}{\partial y}\right).$$

By using (16.14), it can be shown that the following equality holds:

$$\overline{\varphi(z)}+\bar{z}\varphi'(z)+\psi(z)=\frac{1}{2}\left(\frac{\partial w_*}{\partial x}-i\frac{\partial w_*}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial w^1}{\partial x}-i\frac{\partial w^1}{\partial y}\right). \tag{16.19}$$

By transforming to the conjugate values in Eqs. (16.18) and (16.19), we obtain equations identical in form with Eqs. (15.10) and (15.11) for the plane problem in elasticity.

(3) Suppose that values of the deflection and of the bending moment are given at the edge of the plate (so-called simply supported edge). The boundary conditions are

$$\operatorname{Re}\left\{\varkappa_1^*\varphi'(t)-\left(\frac{dt}{ds}\right)^2[t\varphi''(t)+\varphi'(t)]\right\} \quad h(t), \tag{16.20}$$

$$\operatorname{Re}\left\{\frac{dt}{ds}\left[\overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)\right]\right\}=g(t), \tag{16.21}$$

where $\varkappa^*=\frac{2(1+\nu)}{1-\nu}$ and $h(t)$, $g(t)$ are given functions.

Let now the region occupied by the middle surface be multiple connected, i.e., let it be bounded by $m+1$ contours L_n. In this case, too, we have a complete analogy with the plane problem. The functions $\varphi(z)$ and $\psi(z)$ are found to be multiple-valued, and the multiple-valuedness is eliminated by introducing new functions, $\varphi^*(z)$ and $\psi^*(z)$:

$$\begin{gathered}\varphi(z)=\frac{1}{2\pi i}\sum_{k=1}^{m}\left[\frac{iP_{zk}^*}{8D}z+\frac{M_{xk}^*+iM_{yk}^*}{8D}\right]\ln(z-z_k)+\varphi^*(z),\\ \psi(z)=-\frac{1}{2\pi i}\sum_{k=1}^{m}\frac{1}{8D}(M_{xk}^*-iM_{yk}^*)\ln(z-z_k)+\psi^*(z).\end{gathered} \tag{16.22}$$

Here P_{zk}^* is the resultant vector of the forces applied to the contour L_k, and M_{xk}^*, M_{yk}^* are the components of the resultant moment of the forces.

The formulation of the bending problem for piecewise homogeneous bodies is obvious. In contrast to the plane strain problem, this kind of non-homogeneity (just as for plane stress) may be due not only to a change in mechanical properties, but also to an abrupt change in

thickness. The corresponding sets of variants of contact conditions are obtained on the basis of relations (16.18) and (16.19) similarly to (15.16) and (15.17). Naturally, in the case of a change in thickness the effect of stress concentration due to this change is not taken into account.

17. On Singular Solutions of Elastic Equations

The problems of the theory of elasticity reduce to boundary value problems for a certain system of differential equations and therefore, according to the classical formulation, it is natural to consider only solutions that have all derivatives entering into the equations in an open region and all derivatives appearing in the boundary conditions in a closed region. However, the presence of irregular points on the boundary (angular and conic points, edges, vertices of polyhedral angles) and also points where boundary conditions of different kind adjoin accounts for the fact that there exists no solution of problems in the above class. It is therefore necessary to broaden the formulation of the problem by admitting the presence of non-differentiable displacements. Of course, the resulting unboundedness of strains and stresses is at variance with the basic principles of the linear theory of elasticity.* However, the construction of such solutions has proved very useful in a theoretical sense, and the solutions themselves have found various applications. It appears that the solution, as a rule, becomes valid at a small distance from irregular points. Of course, it is possible to avoid the unboundedness of strains and stresses in various ways (by "smoothing" the boundary or by passing to a non-linear problem with the use of any one of the variants of the theory of finite strains), but this proves to be more difficult to realize. Moreover, as special studies on fracture mechanics show (see V. Z. Parton, E. M. Morozov [1]), the coefficients entering into the asymptotic expansion for stresses permit a direct evaluation of strength and fracture.

Before proceeding to the consideration of the basic properties of solutions of elastic equations in the neghbourhood of irregular points of the boundary, we note the following. Singularities in the solutions of elasticity problems may also occur in the case of a smooth boundary when a concentrated force or moment is applied or when the boundary conditions undergo a discontinuity. By superimposing particular solutions, it is possible to modify the boundary conditions so as to remove the irregularity of the solution. In the following discussion the boundary conditions are therefore assumed to be sufficiently smooth.

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* The strains must be sufficiently small for their squares to be negligible compared with unity and for Hooke's law to be valid.

V. A. Kondrat'ev [1] studied the asymptotic behaviour of the solution of two-dimensional problems for elliptic systems of equations (which include elasticity problems) in the neighbourhood of angular points. It is proved that the solution is representable as the sum of an infinitely differentiable function and an asymptotic series each term of which is the solution of the problem for a wedge with the same apex angle subject to homogeneous boundary conditions of the same kind as for the original problem. Naturally, these solutions (they are termed eigensolutions) are found to within factors determined by the boundary value problem as a whole. A similar statement holds for the case of conic points. Here it is necessary to obtain solutions for a cone with homogeneous boundary conditions.

Let there be an edge (angular line) on a surface. It is established (O. K. Aksentyan [1], V. A. Koldorkina [1]) that the asymptotic behaviour of the solution at a particular point of the edge (in a plane perpendicular to it) is the same as in plane strain and antiplane strain. This statement enables us to extend the results of V. A. Kondrat'ev to the three-dimensional case of the kind indicated above.

It should be noted (and this will be verified in what follows) that it is possible to construct an infinite number of eigensolutions for model regions. In problems having a definite physical meaning, however, solutions involving an infinite strain energy are excluded. To simplify computational algorithms, the eigensolutions of principal interest must be those with the strongest singularity. True, in some exceptional cases it may happen that the factor multiplying the solution with the strongest singularity vanishes, and it is then necessary to proceed to the next solution. Such a situation occurs, for example, in twisting a body in the form of a lens.

We now turn to the consideration of eigensolutions for a wedge (see S. N. Karp, F. G. J. Karal [1]). Let the wedge occupy the region $0 \leqslant r < \infty$, $|\theta| < \alpha$ (in polar co-ordinates). Lamé's equations (14.4) for displacements in polar co-ordinates in the case of plane strain ($\varepsilon_z = 0$) are of the form

$$\begin{gathered}
(\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)- \\
-\mu\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\theta}{r}\right)=0, \\
(\lambda+2\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+ \\
+\mu\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\theta}{r}\right)=0.
\end{gathered} \tag{17.1}$$

Below are the expressions for the components of the stress tensor:

$$\sigma_\theta = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right),$$
$$\sigma_r = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + 2\mu\frac{\partial u_r}{\partial r}, \tag{17.2}$$
$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

We proceed from the representation of displacements in the form

$$u_r(r, \theta) = r^\gamma f(\theta), \qquad u_\theta(r, \theta) = r^\gamma g(\theta). \tag{17.3}$$

From (15.1) we obtain

$$\nu_1 f'' + (\gamma^2 - 1) f + [(\gamma - 1) - \nu_1(\gamma + 1)] g' = 0,$$
$$g'' + \nu_1(\gamma^2 - 1) g + [(\gamma + 1) - \nu_1(\gamma - 1)] f' = 0.$$

where $\nu_1 = (1 - 2\nu)/2(1 - \nu)$.

The stresses and displacements are then expressed as

$$\mu^{-1} r^{1-\gamma}\sigma_\theta = -2\gamma A\cos[(1+\gamma)\theta] - 2\gamma B\sin[(1+\gamma)\theta] -$$
$$-(1+\gamma)(1-\nu_2) C\cos[(1-\gamma)\theta] - (1+\gamma)(1-\nu_2) D\sin[(1-\gamma)\theta],$$
$$\mu^{-1} r^{1-\gamma}\sigma_r = 2\gamma A\cos[(1+\gamma)\theta] + 2\gamma B\sin[(1+\gamma)\theta] +$$
$$+ 2C\left\{\nu\frac{1+\gamma-(1-\gamma)\nu_2}{1-2\nu} + \gamma\right\}\cos[(1-\gamma)\theta] +$$
$$+ 2D\left\{\nu\frac{1+\gamma-(1-\gamma)\nu_2}{1-2\nu} + \gamma\right\}\sin[(1-\gamma)\theta], \tag{17.4}$$
$$\mu^{-1} r^{1-\gamma}\tau_{r\theta} = -2\gamma A\sin[(1+\gamma)\theta] + 2\gamma B\cos[(1+\gamma)\theta] -$$
$$-(1-\gamma)(1-\nu_2) C\sin[(1-\gamma)\theta] + (1-\gamma)(1-\nu_2) D\cos[(1-\gamma)\theta],$$
$$r^{-\gamma} u_r = A\cos[(1+\gamma)\theta] + B\sin[(1+\gamma)\theta] +$$
$$+ C\cos[(1-\gamma)\theta] + D\sin[(1-\gamma)\theta],$$
$$r^{-\gamma} u_\theta = B\cos[(1+\gamma)\theta] - A\sin[(1+\gamma)\theta] + \nu_2 D\cos[(1-\gamma)\theta] -$$
$$-\nu_2 C\sin[(1-\gamma)\theta],$$

where A, B, C, D are arbitrary constants, $\nu_2 = (4\nu - 3 - \gamma)/4\nu - 3 + \gamma$.

Consider the case when stresses are prescribed on each side of the wedge (II-II). The values of σ_θ and $\tau_{r\theta}$ for $\theta = \pm\alpha$ must be equated to zero. As a result, we arrive at a homogeneous system of equations, which falls into two systems of the second order:

$$-2\gamma A\cos[(1+\gamma)\alpha] - (1+\gamma)(1-\nu_2) C\cos[(1-\gamma)\alpha] = 0,$$
$$2\gamma A\sin[(1+\gamma)\alpha] - (1-\gamma)(1-\nu_2) C\sin[(1-\gamma)\alpha] = 0. \tag{17.5}$$

$$2\gamma B\sin[(1+\gamma)\alpha]\,(1+\gamma)(1-\nu_2) D\sin[(1-\gamma)\alpha] = 0,$$
$$2\gamma B\cos[(1+\gamma)\alpha] + (1-\gamma)(1-\nu_2) D\cos[(1-\gamma)\alpha] = 0. \tag{17.6}$$

The condition for the existence of a non-trivial solution of system (17.5) leads to the equation

$$\sin 2\alpha + \frac{1}{\gamma} \sin 2\alpha\gamma = 0. \tag{17.7}$$

Hence, if γ is its root, we arrive at an eigensolution for the wedge for which $B = D = 0$ and the constants A and C are related by the equality

$$A = \frac{(1+\gamma)(\nu_2 - 1) \cos [(1-\gamma)\alpha]}{2\gamma \cos [(1+\gamma)\alpha]} C = k_1 C. \tag{17.8}$$

The solution is thus expressed in terms of a single constant.

The case when the determinant of system (17.8) is zero is considered in a similar way. The equation is then of the form

$$\sin 2\alpha - \frac{1}{\gamma} \sin 2\alpha\gamma = 0.$$

Consequently, it may be stated that the non-trivial solution of the problem (II-II) for a wedge exists when the exponent satisfies the equation

$$\sin 2\alpha\gamma = \pm \sin 2\alpha. \tag{17.9}$$

Other boundary value problems (including those where conditions of different types are prescribed on the sides of a wedge) are treated in a similar manner. Below are given the corresponding equations:

$$\sin 2\alpha\gamma = \pm \frac{\alpha}{\varkappa} \sin 2\alpha \quad \text{(I-I)}, \tag{17.10}$$

$$\sin 2\alpha\gamma = \pm \sin 2\alpha \quad \text{(III-III)}, \tag{17.11}$$

$$\sin^2 2\alpha\gamma = \frac{(1+\varkappa)^2}{4\varkappa} - \frac{\gamma^2 \sin^2 2\alpha}{\varkappa} \quad \text{(I-II)}, \tag{17.12}$$

$$\sin 4\alpha\gamma = -\gamma \sin 4\alpha \quad \text{(II-III)}, \tag{17.13}$$

$$\sin 4\alpha\gamma = \frac{\gamma}{\varkappa} \sin 4\alpha \quad \text{(I-III)}, \tag{17.14}$$

where $\varkappa = (3 - \nu)/(1 + \nu)$.

Here III designates a problem where the shearing stress and the normal component of the displacement on the side of a wedge are assumed to be zero.

The foregoing equations have, in general, complex (and necessarily conjugate) roots $\gamma = \gamma_1 + i\gamma_2$; this leads, in transforming to the actual representation, to eigenfunctions of the form

$$r^{\gamma_1} \begin{cases} \sin (\ln \gamma_2 r) \\ \cos (\ln \gamma_2 r). \end{cases} \tag{17.15}$$

It follows from (17.15) that the stress singularity is of an oscillation nature. As some solutions show (see N. I. Muskhelishvili [3], p. 417),

the oscillation is significant only over a very small distance from an angular point, which allows the oscillating factor to be neglected in design schemes. Figure 7 (taken from the work of A. I. Kalandiya [3]) gives values of min Re $\gamma > 0$ for all combinations of boundary conditions considered above when $\nu = 0.3$.

The foregoing procedure can be extended to the case when there is a conic point (see V. A. Kondrat'ev [1]). For simplicity, we construct solutions for the case of a circular cone (see Z. P. Bazant,

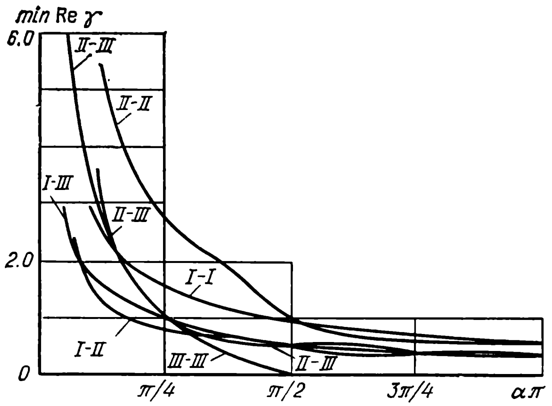

Fig. 7. Relation between min Re γ and the angle α for different boundary conditions.

L. N. Keer [1]). Let α be the vertex angle. The displacements are represented in a spherical co-ordinate system as

$$u_r = r^\gamma U_r(\theta), \quad u_\theta = r^\gamma U_\theta(\theta), \qquad u_\varphi = 0. \tag{17.16}$$

The resulting system of equations for U_r and U_θ is then

$$\begin{gathered}\frac{d^2U_r}{d\theta^2} + \cot\frac{dU_r}{d\theta} + [\nu_3(\gamma-1)-\gamma-1]\frac{dU_\theta}{d\theta} + \\ + \nu_3(\gamma-1)(\gamma+2)U_r + [\nu_3(\gamma-1)-\gamma-1]\cot\theta\, U_\theta = 0, \\ \nu_3\frac{d^2U_\theta}{d\theta^2} + [\nu_3(\nu+2)-\gamma]\frac{dU_r}{d\theta} + \nu_3\cot\frac{dU_\theta}{d\theta} + \\ + \left[\gamma(\gamma+1) - \frac{\nu_3}{\sin^2\theta}\right]U_\theta = 0,\end{gathered} \tag{17.17}$$

where $\nu_3 = (1-\nu)/(0.5-\nu)$.

The boundary conditions in the case of the first problem are

$$U_r(\alpha) = U_\theta(\alpha) = 0$$

and in the case of the second problem

$$\sigma_\theta = \frac{dU_\theta}{d\theta} + (1 + \nu_1 + \gamma\nu_1) U_r + \nu_1 \cot \alpha U_\theta = 0 \quad \text{for} \quad \theta = \alpha,$$

$$\tau_{r\theta} = \frac{dU_r}{d\theta} - (1 - \nu) U_\theta = 0 \quad \text{for} \quad \theta = \alpha$$

Moreover, from the regularity condition for the solution for $\theta = 0$ it follows that

$$\frac{dU_r}{d\theta} = U_\theta = 0.$$

Figure 8 gives values of min Re γ as a function of α for several values of ν for the first and second boundary value problems.

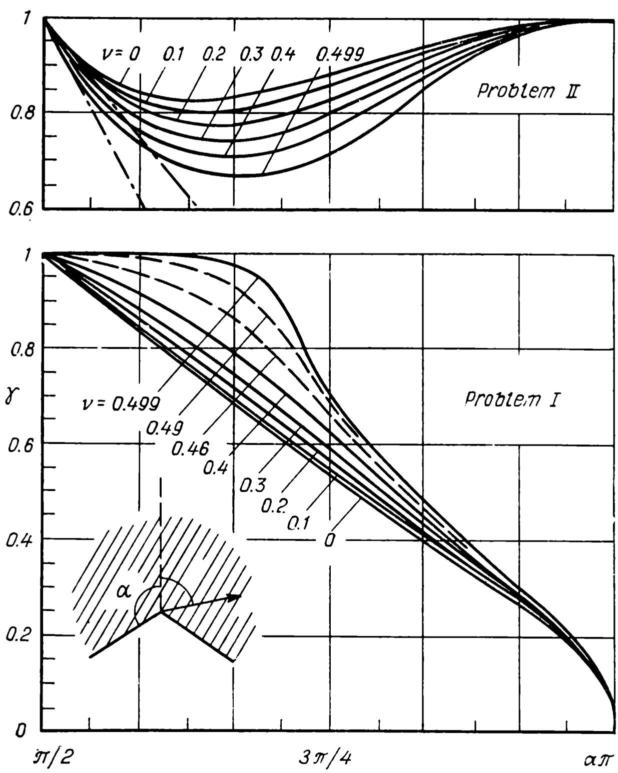

Fig. 8. Relation between Re γ and the angle α for a conic point

Note, also, a very simple case arising in the torsion problem. The solution of the Dirichlet problem for a wedge with zero conditions

is in this case

$$r^{\frac{\pi}{2\alpha}} \cos\left(\frac{\pi}{2\alpha}\theta\right). \tag{17.18}$$

Consequently, the infinite torsional stresses occur only for an angle α greater than $\pi/2$.

As follows from the foregoing, the singularity index in the neighbourhood of irregular points of the boundary can be established without regard to the solution of the boundary value problem itself, but the factors multiplying the eigensolutions can, in general, be found from all the data of the problem (the configuration of the region and the boundary conditions). The procedures for finding these factors will be discussed in Secs. 25 and 37.

The foregoing procedure was subsequently extended to piecewise homogeneous wedges when the interface between media is a straight line passing through the vertex. It was assumed that there was cohesion on the lines of contact between the media. The solution for each wedge is represented, as before, in the form of (17.3) (or in a similar form if the real variable technique is used). The condition of equality of displacements and stresses at the interfaces between the media and the conditions on the outer sides furnish a transcendental equation for the parameter γ.

M. L. Williams [2] has obtained the solution of the problem for two wedges of angles π. This problem corresponds to a piecewise homogeneous plane with a cut along the interface. In another work (see A. R. Zak, M. L. Williams [1]) the solution is constructed for three wedges: two wedges of angle $\pi/2$ of the same material and one wedge of angle π of dissimilar material. This problem corresponds to a piecewise homogeneous plane with a cut in one of the half-planes approaching the interface between the media. An analysis of solutions of problems for a body composed of two wedges with zero displacements on the outer boundaries is given by A. G. Avetisyan and K. S. Chobanyan [1].

Chapter IV

INTEGRAL EQUATIONS FOR TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY

18. Muskhelishvili's Integral Equations

Consider, simultaneously, the first and second problems of the theory of elasticity for a finite simply connected region D^+ bounded by a smooth contour L. The boundary conditions are rewritten in a unified form:

$$k\overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)=\overline{f(t)}. \tag{18.1}$$

Here $k = -\varkappa$ in the case of the first problem, $k = 1$ in the case of the second problem, and the function $f(t)$ has been defined above (Sec. 15).

Condition (18.1) is rewritten as

$$\psi(t)=\overline{f(t)}-k\overline{\varphi(t)}-\bar{t}\varphi'(t). \tag{18.1$'$}$$

The right-hand side of (18.1′) is thus a boundary value of a function analytic in the region D^+. According to (2.9′), this condition can be represented as an equality valid for all points z outside the region D^+:

$$\frac{1}{2\pi i}\int\limits_L \frac{k\overline{\varphi(t)}+\bar{t}\varphi'(t)}{t-z}\,dt=\frac{1}{2\pi i}\int\limits_L \frac{\overline{f(t)}\,dt}{t-z}=A(z). \tag{18.2}$$

Let us transform from the functional equation (18.2) to an integral one. To do this, we pass to the points of the contour L by a limiting process remaining all the time outside the region D^+. In so doing we assume that the function $\varphi(t)$, $\varphi'(t)$, and $f(t)$ satisfy the H-L condition, and it is therefore possible to use the Sokhotskii-Plemelj formulas.

To obtain an equation in compact form, we take advantage of the identities

$$-\frac{1}{2}\overline{\varphi(t_0)}+\frac{1}{2\pi i}\int\limits_L \frac{\overline{\varphi(t)}\,\overline{dt}}{\bar{t}-\bar{t}_0}=0,$$

$$-\frac{1}{2}\varphi'(t_0)+\frac{1}{2\pi i}\int\limits_L \frac{\varphi'(t)\,dt}{t-t_0}=0.$$

We apply a limiting process in (18.2) and add to this relation the last identities multiplied, respectively, by $\bar{t}_0$ and k. By performing the integration by parts, we arrive at an *integral equation* due to *N. I. Muskhelishvili* [3]:

$$-k\overline{\varphi(t_0)}-\frac{k}{2\pi i}\int_L \overline{\varphi(t)}\, d\ln\frac{\bar{t}-\bar{t}_0}{t-t_0}-\frac{1}{2\pi i}\int_L \varphi(t)\, d\frac{\bar{t}-\bar{t}_0}{t-t_0}=A(t_0). \tag{18.3}$$

Equation (18.3) belongs to the class of Fredholm integral equations. Let us make its analysis. We begin our consideration with the second fundamental problem ($k=1$). It will first be shown that every solution of this equation must be a boundary value of a function analytic in the region D^+. Let $\varphi(t)$ be any solution of Eq. (18.3). We form Cauchy-type integrals (the point z' is taken in the region D^-):

$$\begin{gathered}\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z'}\, dt=i\Phi(z'),\\ \frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)}+\bar{t}\varphi'(t)-\overline{f(t)}}{t-z'}\, dt=-i\Psi(z').\end{gathered} \tag{18.4}$$

Equation (18.3) may then be interpreted as a relation between functions $\Phi(t)$ and $\Psi(t)$, which are boundary values of analytic functions in D^-:

$$\overline{\Phi(t)}+\bar{t}\Phi'(t)+\Psi(t)=0. \tag{18.5}$$

It follows from relation (18.5) that

$$\Phi(z')=i\alpha z'+\beta, \quad \Psi(z')=-\bar{\beta},$$

since these functions are the solution of the second exterior problem with zero stress values on the contour (α is a real constant, β is a complex constant). Since the same functions are Cauchy-type integrals, they are zero at infinity. It follows, therefore, from the first representation of (18.4) that $\varphi(t)$ is a boundary value of a function analytic in the region D^+.

Since the function $\varphi_0(z)=i\varphi z+\beta$ (α and β are, as before, a real and a complex constant) corresponds to a zero state of stress, it is obvious that this function is a nontrivial (and unique) solution of Eq. (18.3) with a zero right-hand side.

Note that the elasticity problem itself for a bounded region has a solution only when the resultant moment of the external forces is zero:

$$\operatorname{Re}\int_L \overline{f(t)}\, dt=0. \tag{18.6}$$

The condition that the resultant vector of the external forces is zero automatically follows from the uniqueness of the boundary condition.

Following D. I. Sherman [2], we prove that condition (18.6) ensures the solvability of Eq. (18.3) (when $k = 1$). Suppose that the origin of co-ordinates is situated in the region D^+. To the left-hand side of the original equation we add the operator

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt + \frac{1}{2\pi i}\frac{1}{t_0}\int_L\left[\frac{\varphi(t)}{t^2}\,dt + \frac{\overline{\varphi(t)}}{\bar{t}^2}\,d\bar{t}\right] \tag{18.7}$$

and examine the resulting equation

$$-k\overline{\varphi(t_0)} - \frac{k}{2\pi i}\int_L \overline{\varphi(t)}\,d\ln\frac{\bar{t}-\bar{t}_0}{t-t_0} - \frac{1}{2\pi i}\int_L \varphi(t)\,d\,\frac{\bar{t}-\bar{t}_0}{t-t_0} +$$

$$+\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt + \frac{1}{2\pi i}\frac{1}{t_0}\int_L\left[\frac{\varphi(t)}{t^2}\,dt + \frac{\overline{\varphi(t)}}{\bar{t}^2}\,d\bar{t}\right] = A(t_0). \tag{18.3'}$$

We prove that every solution of the last equation is a boundary value of an analytic function in the region D^+. In contrast to (18.4), the corresponding function $\Psi(z')$ is defined as follows:

$$\Psi(z') = \frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)} + \bar{t}\varphi'(t) - \overline{f(t)}}{t - z'}\,dt + \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt +$$

$$+\frac{1}{2\pi i}\frac{1}{z'}\int_L\left[\frac{\varphi(t)}{t^2}\,dt + \frac{\overline{\varphi(t)}}{\bar{t}^2}\,d\bar{t}\right]. \tag{18.4'}$$

Equation (18.3') is then transformed into the same limiting relation (18.5). Similarly to the foregoing, we find that the functions $\Phi(z')$ and $\Psi(z')$ are zero and hence the solution can be continued into the region D^+.

Let us now show that every solution of Eq. (18.3') makes the additional terms (18.7) zero if the right-hand side satisfies condition (18.6). Since the function $\Psi(z')$, is identically zero, it follows that the coefficients in its expansion in a Laurent series are also zero. The first coefficient in the expansion is

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt = 0, \tag{18.8}$$

and the coefficient of $1/z'$ is

$$\frac{1}{2\pi i}\int_L\left[\overline{\varphi(t)} + \bar{t}\varphi'(t)\right]dt -$$

$$-\frac{1}{2\pi i}\int \overline{f(t)}\,dt + \frac{1}{2\pi i}\int_L\left[\frac{\varphi(t)}{t^2}\,dt + \frac{\overline{\varphi(t)}}{\bar{t}^2}\,d\bar{t}\right] = 0. \tag{18.9}$$

It follows from (18.8) that the first integral in (18.7) always vanishes. We rearrange (18.9) in the form

$$\frac{1}{2\pi i}\int_L \left[\overline{\varphi(t)}\,dt-\varphi(t)\,d\bar{t}\,\right]-\frac{1}{2\pi i}\int_L \bar{f}(t)\,dt+$$
$$+\frac{1}{2\pi i}\int_L \left[\frac{\varphi(t)}{t^2}\,dt+\frac{\overline{\varphi(t)}}{\bar{t}^2}\,d\bar{t}\right]=0. \tag{18.10}$$

The first term, as well as the second one, by (18.6), is real, and the third term is imaginary. Consequently, the third term in (18.7) is zero.

Let us prove that Eq. (18.3′) is solvable for any right-hand side. To do this, we must show that the homogeneous equation has no non-trivial solutions. Suppose that such a solution exists. Denote it by $\varphi_0(t)$. Since the right-hand side of the equation is zero, condition (18.6) is automatically fulfilled, and hence sum (18.7) becomes zero when the function $\varphi_0(t)$ is substituted in it. Consequently, the function $\varphi_0(t)$ must also be a solution of the homogeneous equation (18.3), and it is therefore equal to $\alpha t+\beta$. Substituting this function in (18.7), we find from the condition of each term being zero that the constants α and β vanish. This proves the solvability of Eq. (18.3′) with an arbitrary right-hand side, and in particular when condition (18.6) is fulfilled. In the latter case, as has been proved above, the solution of Eq. (18.3′) is also a solution of Eq. (18.3).

Muskhelishvili's integral equations can be constructed for the case of a multiply connected region and an exterior problem. D. I. Sherman [2] has made an analysis of these equations and proved their solvability. However, the actual implementation of solutions in these cases is difficult because of the necessity for first solving auxiliary problems for some particular types of loading.*

D. I. Sherman has established that the characteristic numbers of Eq. (18.3′) are greater than unity in modulus, and this ensures the convergence of the method of successive approximations. Naturally, this conclusion also holds for Muskhelishvili's integral equation (of course, provided the condition for zero resultant moment of the external forces is fulfilled).

The numerical solution of Muskhelishvili's equation by the mechanical quadrature method is difficult because of the presence of an eigenfunction since the determinant of the corresponding system of linear equations is zero (to the present approximation), which ac-

* From the results of the work of D. I. Sherman (The Static Plane Problem of the Theory of Elasticity for an Anisotropic Medium, *Trudy Seismologicheskogo Inst. Akad. Nauk SSSR*, No. 86, 1938), applied to an isotropic medium one can obtain always solvable integral equations differing from Muskhelishvili's equations by additional operators of an elementary kind.

counts for unstable values of the unknowns. To eliminate this defect, N. I. Muskhelishvili [1] and also A. Ya. Gorgidze and A. K. Rukhadze [1] proposed that the density function should be fixed at some points, excluding the consideration of the corresponding equations. P. I. Perlin and Yu. N. Shalyukhin [1] suggested a different procedure. The authors consider Eq. (18.3′). Because of the error inherent in the quadrature formulas used and because of the piecewise constant (or some other) representation of the density function, the terms included additionally are different from zero, but the error introduced by them is small (of the order of the error involved in using the quadrature formulas); however, the structure of the system of algebraic equations is radically improved since the equation in question has no eigenfunctions.

Note that the solution of Muskhelishvili's equation by the method of successive approximations also involves complications of a similar kind (because of the error in numerical implementation) (see Sec. 12). P. I. Perlin and Yu. N. Shalyukhin also suggested that Eq. (18.3′) should be used to construct a convergent solution.

Attention is drawn to the fact that when there are two axes of symmetry in problems, the process in both approaches is stable if, of course, the discretization of the contour necessary for numerical implementation has the corresponding symmetry.

The investigation of the integral equation corresponding to the first fundamental problem is conducted in a simple way. In this case it is necessary to introduce an additional functional

$$\frac{1}{2\pi i}\int\limits_{L}\frac{\varphi(t)}{t}\,dt.$$

19. Sherman-Lauricella Integral Equations

We now turn to the construction of other integral equations for the fundamental plane problems of elasticity known as the *Sherman-Lauricella equations* (see D. I. Sherman [6, 7]).

Let a region D be bounded by one or several contours, $L_1, L_2, \ldots, L_m, L_0$, where the first m contours are situated outside each other, and the last one contains all the others (the contour L_0 may be absent). Let the finite regions bounded by the contours L_k be denoted by D_k^+ $(k = 1, 2, \ldots, m)$, and the infinite region, i.e., the outside of the contour L_0, by D_0^-.

We begin the consideration with the second fundamental problem. Suppose that the resultant vectors X_k, Y_k of the external forces applied to the contours L_k are zero (otherwise it would be necessary to make an obvious transformation of the boundary conditions). The unknown functions $\varphi(z)$ and $\psi(z)$ are then single valued.

The boundary conditions are written as ($L = L_k + L_0$)

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C_k \quad (t \in L). \tag{19.1}$$

Here $f(t)$ is a given function, single valued and continuous on each of the contours, the constants C_0 and C_k are determined in solving the boundary value problem. One of them, say C_{m+1}, may be fixed arbitrarily by setting it equal to zero. Naturally, it is assumed that conditions (18.6) are fulfilled.

The functions $\varphi(z)$ and $\psi(z)$ are sought in the form*

$$\varphi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)\,dt}{t-z}, \tag{19.2}$$

$$\psi(z) = \frac{1}{2\pi i}\int_L \frac{\overline{\omega(t)}}{t-z}\,dt + \frac{1}{2\pi i}\int_L \frac{\omega(t)}{t-z}\,d\bar{t} - \\ - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega(t)\,dt}{(t-z)^2} + \sum_{k=1}^{m} \frac{b_k}{z-z_k}. \tag{19.3}$$

Here $\omega(t)$ is the unknown function, z_k are arbitrarily prescribed points in the regions D_k^+, b_k are real constants defined as follows:

$$b_k = i\int_{L_k} [\omega(t)\,dt - \overline{\omega(t)\,dt}]. \tag{19.4}$$

Note that representations (19.2) and (19.3) resemble in form similar representations of the solution of the problem for a half-plane.

Suppose that the function $\omega(t)$ satisfies the H-L condition. By applying a limiting process in (19.2) and (19.3), as well as in a representation for the function $\varphi'(z)$ obtained from (19.2), we pass to the points of the contours L_k from the outside, and to the points of the contour L_0 from the inside. Substituting the resulting expressions in the boundary conditions (19.1), we obtain a Fredholm integral equation, namely

$$\omega(t_0) + \frac{1}{2\pi i}\int_L \omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0} + \\ + \sum_{k=0}^{m} \frac{b_k}{\bar{t}_0-\bar{z}_k} = C_k + f(t) \quad (t \in L). \tag{19.5}$$

* In the works of D. I. Sherman [6, 7] the expression for $\varphi(z)$ contains additional terms for which there is no special need (this was pointed out by the author later).

The left-hand side of this expression incorporates the term

$$\frac{\bar{b}_0}{\bar{t}_0 - z_0}, \tag{19.6}$$

$$b_0 = \frac{1}{2\pi i} \int_L \left[\frac{\omega(t)}{t^2}\, dt + \frac{\overline{\omega(t)}}{\bar{t}^2}\, d\bar{t} \right], \tag{19.7}$$

and $z_0 = 0$ has been included for uniformity in writing.

Let the constants C_k be defined by relations of the form

$$C_k = -\int_{L_k} \omega(t)\, ds, \tag{19.8}$$

where ds is a differential arc length of the contour. With this equation, (19.5) transforms into an integral equation for $\omega(t)$.

Let us show that every solution of Eq. (19.5) makes the number b_0 zero if the condition for zero resultant moment of the forces applied to the whole body is fulfilled. To prove this, Eq. (19.5) is represented as

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \frac{\bar{b}_0}{\bar{t}} - C_j = f(t) \quad (t \in L). \tag{19.9}$$

We multiply both sides of this equality by $d\bar{t}$ and integrate. Since the functions $\varphi(t)$ and $\psi(t)$ are single valued, we obtain

$$\int_L \left[\varphi(t)\, d\bar{t} - \overline{\varphi(t)}\, dt\right] - 2\pi i \bar{b}_0 = \int_L f(t)\, d\bar{t}. \tag{19.10}$$

Since all terms in this equality, except $2\pi i \bar{b}_0$, are imaginary, it follows that $b_0 = 0$.

Let us prove that Eq. (19.5) is always solvable. Assume, as usual, that there is a non-trivial solution $\omega_0(t)$ of the homogeneous equation. The corresponding functions $\varphi(z)$ and $\psi(z)$ are furnished with the subscript zero, and the constants b_k and C_k defined by relations (19.2) to (19.4) and (19.8) with the superscript zero. The functions $\varphi_0(z)$ and $\psi_0(z)$ must satisfy the boundary conditions

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} - C_k^0 = 0 \quad (t \in L). \tag{19.11}$$

The uniqueness theorem gives

$$\varphi_0(z) = i\alpha z + \beta, \quad \psi_0(z) = -\bar{\beta}, \quad C_k^0 = 0. \tag{19.12}$$

Referring to formulas (19.2) and (19.3), and noting (19.12), we arrive at the equalities

$$i\alpha z + \beta = \frac{1}{2\pi i} \int_L \frac{\omega_0(t)\, dt}{t - z},$$

$$-\bar{\beta} = \frac{1}{2\pi i} \int_L \frac{\overline{\omega_0(t)}\, dt}{t - z} - \frac{1}{2\pi i} \int_L \frac{\bar{t}\omega_0'(t)}{t - z}\, dt + \sum_{k=1}^{m} \frac{b_k}{z - z_k}. \tag{19.13}$$

We now introduce, on all the contours $L_k + L_0$, the functions

$$i\varphi^*(t) = \omega_0(t) - i\alpha t - \beta, \tag{19.14}$$

$$i\psi_-^*(t) = \overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum_{k=1}^{m} \frac{b_k}{t - z_k}. \tag{19.15}$$

Since these functions are analytic in the regions D_k^+, equalities (19.13) may be rewritten as

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)}{t-z}\,dt = 0, \quad \frac{1}{2\pi i}\int_L \frac{\psi^*(t)}{t-z}\,dt = 0, \quad (z \in D). \tag{19.16}$$

It follows from the property of the Cauchy-type integral (Sec. 2) that the functions $\varphi^*(t)$ and $\psi^*(t)$ are boundary values of functions analytic in the regions D_k^+ and D_0^- with $\varphi^*(\infty) = 0$ and $\psi^*(\infty) = 0$.

It has been previously shown that $b_0 = 0$ if condition (18.6) is fulfilled. It is obvious that in our case this condition is also fulfilled and therefore we have $b_0^0 = 0$. Substituting the expression for $\omega_0(t)$ from (19.14) on the right-hand side of (19.7), we arrive at the equality $\alpha = 0$. By further eliminating the function $\omega_0(t)$ from (19.14) and (19.15), we obtain

$$\overline{\varphi^*(t)} + t\varphi^{*\prime}(t) + \psi^*(t) = i\sum_{k=1}^{m} \frac{b_k^0}{t - z_k} - 2i\beta. \tag{19.17}$$

By multiplying both sides of this equality by dt, and integrating over each contour L_k, we arrive at the equalities

$$\int_{L_k} \left[\overline{\varphi^*(t)}\,dt - \varphi^*(t)\,dt\right] = -2\pi b_k^0, \tag{19.18}$$

from which it follows that

$$b_k^0 = 0. \tag{19.19}$$

We therefore obtain

$$\overline{\varphi^*(t)} + \bar{t}\varphi^*(t)\psi + \psi^*(t) = -2i\bar{\beta} \quad (t \in L_k).$$

Thus, the functions $\varphi^*(z)$ and $\psi^*(z)$ solve the second fundamental problem for each of the regions D_k^+ with zero values on the boundary. By the uniqueness theorem for the region D_0^- [noting that $\varphi^*(\infty) = \psi^*(\infty) = 0$ and hence $\varphi^*(z) = \psi^*(z) = 0$], we find that $= 0$. Then

$$\varphi^*(z) = i\alpha_k z + \beta_k, \quad \psi^*(z) = -\beta_k \quad (z \in D_k^+).$$

From this, on the basis of formulas (19.14), (19.15), it follows that

$$\omega_0(t) = -\alpha_k t + i\beta_k \quad (t \in L_k),$$
$$\omega_0(t) = 0 \quad (t \in L_0).$$

Now, by using (19.19) and the equalities $C_k^0 = 0$, we find that all α_k and β_k are zero. Hence, $\omega_0(t) \equiv 0$. Thus, Eq. (19.5) is always solvable. If the resultant moment is zero, $b_0 = 0$, and the solution of Eq. (19.5) is therefore identical with that of the original integral equation. Consequently, this equation is always solvable.

We next consider the first fundamental problem. The boundary condition is

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = f(t) \quad (t \in L). \tag{19.20}$$

In this case the representations for the functions $\varphi(z)$ and $\psi(z)$ are chosen as

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)}{t-z}\,dt + \sum_{k=1}^{m} A_k \ln(z-z_k), \tag{19.21}$$

$$\psi(z) = -\frac{\varkappa}{2\pi i}\int\limits_L \frac{\overline{\omega(t)}}{t-z}\,dt + \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)}{t-z}\,d\bar{t} - \frac{1}{2\pi i}\int\limits_L \frac{\bar{t}\omega(t)}{(t-z)^2}\,dt - \sum_{k=1}^{m} \varkappa\bar{A}_k \ln(z-z_k), \tag{19.22}$$

where A_k are constants to be determined. We express them in terms of the function $\omega(t)$:

$$A_k = \int\limits_{L_k} \omega(t)\,ds. \tag{19.23}$$

We pass to contour points in the expressions for $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ by a limiting process. After some manipulation we obtain the following integral equation:

$$\varkappa\omega(t_0) + \frac{\varkappa}{2\pi i}\int\limits_L \omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} + \frac{1}{2\pi i}\int\limits_L \overline{\omega(t)}\,d\frac{t-t_0}{\bar{t}-\bar{t}_0} + 2\sum_{k=1}^{m} \varkappa \ln|t-z_j| \int\limits_{L_k} \omega(t)\,ds = f(t) \quad (t \in L). \tag{19.24}$$

The solvability of Eq. (19.24) can be proved by much the same argument as used above.

In the case when the contour L_0 is absent, the argument (with slight modifications) holds.

An integral equation similar in structure to the Sherman-Lauricella equations for the second fundamental problem has been obtained by I. Kh. Khatsirevich [1]. The right-hand side of this equation is expressed directly in terms of the external stresses.

20. Sherman-Lauricella Integral Equations (Continued)

The integral equations constructed and investigated in the preceding section are applicable for the case of arbitrary multiply connected regions. Below is given a modification of these equations taking account of the specific features of regions and the nature of boundary conditions and making it considerably easier to obtain solutions.

For a problem where the region itself and the boundary conditions have mirror and cyclic symmetry of some order, V. M. Buivol

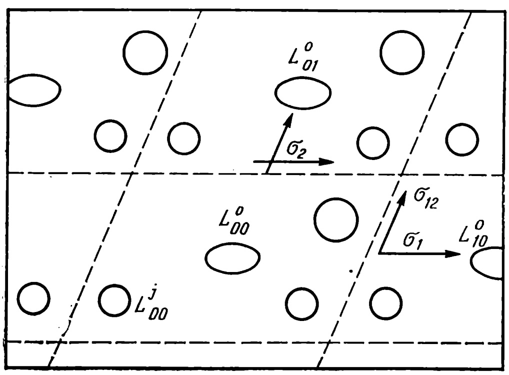

Fig. 9. Parallelogram of periods in a plane with doubly periodic arrangement of holes

[1] has made appropriate transformations of Eq. (19.5) and obtained equations in which the contour of integration is an irreducible part of the contour.*

Let us consider a doubly periodic problem following L. A. Fil'shtinskii [1], who has treated it in the most general formulation. Let there be a parallelogram of periods (Fig. 9), where ω_1 and ω_2 are the primitive periods of the lattice, $\operatorname{Im}\,\omega_1 = 0$, $\operatorname{Im}\,(\omega_2/\omega_1) = 0$, $\omega_2 = ce^{i\alpha}$, $\operatorname{Im}\,c = 0$, let there be a group of l non-overlapping holes inside each of the parallelograms, denoted by L^k_{mn}, where the superscript corresponds to the numbering adopted within the primitive parallelogram, and the subscripts refer to the respective parallelogram.

In formulating the second fundamental problem it must be assumed that the load applied to each of the contours is independent of the subscripts, the resultant vector and the resultant moment being zero in all cases (otherwise the periodicity is violated). Moreover, the formulation of the problem must include the so-called mean values of stresses σ_1, σ_2, and σ_{12} applied on the sides of the paral-

* Some corrections have been introduced by S. B. Vigdergauz [1].

lelogram of periods and equivalent to a uniform state of stress at infinity (with the same stress values).

The representations used for the functions $\varphi(z)$ and $\psi(z)$ are

$$\varphi(z)=\frac{1}{2\pi i}\int_L \omega(t)\,[\zeta(t-z)-\zeta(t)]\,dt+\sum_{k=1}^{l} b_k\zeta(z-z_k)+Az, \quad (20.1)$$

$$\psi(z)=\frac{1}{2\pi i}\int_L [\overline{\omega(t)}\,dt+\omega(t)\,d\bar{t}]\,[\zeta(t-z)-\zeta(t)]-$$
$$-\frac{1}{2\pi i}\int_L \omega(t)\,[\bar{t}\rho(t-z)-\rho_1(t-z)]\,dt+$$
$$+\sum_{k=1}^{l} b_k\,[\zeta(z-z_k)+\rho_1(z-z_k)]+Bz, \quad (20.2)$$

where $\zeta(z)$ is Weierstrass' zeta function, $\rho(z)$ is a Weierstrass elliptic function, the points z_k are situated in the regions D_{00}^{k+}, the constants b_k are determined by formulas similar to (19.4), A and B are constants expressed in terms of the mean stress values and the lattice parameters. The function $\rho_1(z)$ is of the form (see V. Ya. Natanzon [1])

$$\rho_1(z)=\sum_{mn}{}' \left\{\frac{p}{(z-p)^2}-2z\frac{\bar{p}}{p^3}-\frac{\bar{p}}{p^2}\right\} \quad (p=m\omega_1+n\omega_2). \quad (20.3)$$

The corresponding integral equation has been obtained and investigated in the work of L. A. Fil'shtinskii cited above.

Note that V. P. Toropina [1] has considered a periodic problem for a plane with holes. The author has mapped the strip of periods onto a plane with a cut and, using a representation for the functions $\varphi(z)$ and $\psi(z)$ of the type of (19.21) and (19.22), has obtained the corresponding integral equations. From an analysis of the spectral properties it follows that these equations can be solved by the method of successive approximations.

Consider now the question of the numerical solution of the Sherman-Lauricella integral equations and their modifications. These equations are regular, and their solutions can therefore be carried out directly by the mechanical quadrature method (see Sec. 11) taking into account the additional term (19.6). To improve the accuracy, use must be made of the fact that the integrals of the kernels appearing in the Sherman-Lauricella equations are evaluated in closed form. Each term in the integral sums can therefore be represented as the product of the value of the density function at a pivotal point and the corresponding expression.

We now turn to the determination of stresses in an elastic body after finding the solution of the integral equation. For interior points,

this problem is solved in an elementary way since the representations for the functions $\varphi(z)$ and $\psi(z)$, (19.2) and (19.3), are regular. However, the points of principal interest, and at the same time involving the greatest difficulties particularly because of the determination of the function $\varphi'(t)$ are those of the contour.

K. I. Zapparov and P. I. Perlin [1] used the techniques of the spline theory to solve this problem (see Sec. 9). Let the pivotal points be denoted by t_k^0, and the nodal points by t_k. The representation for the function $\varphi(z)$, for example, is then

$$\varphi(t_k^0) = \frac{1}{2}\omega(t_k^0) + \sum_{\substack{j=1\\ j\neq k}}^{N} \omega(t_j^0) \ln \frac{t_{j+1} - t_k^0}{t_j - t_k^0}, \tag{20.4}$$

We construct a cubic spline in each interval t_{k-1}^0, t_k^0:

$$\tilde{\omega}(t) = \sum_{m=0}^{3} a_k^m (t_k^0 - t)^m.$$

The derivative $\tilde{\omega}'(t)$ at the point t_k^0 is then equal to $-a_k^1$.

For the function $\varphi'(t)$ we have the following representation:

$$\varphi'(t_k^0) = \frac{1}{2}\omega'(t_k^0) + \sum_{\substack{j=1\\ j\neq k}}^{N} a_j^1 \ln \frac{t_{j+1} - t_k^0}{t_j - t_k^0}. \tag{20.5}$$

Expression (20.5) enables one to determine the tangential component of stress at the points of the contour according to the formula $\sigma_t = 4\,\mathrm{Re}\,\varphi'(t) - \mathrm{Im}\, f'(t)$.

Figure 10 shows the results of the calculations of σ_t for the outside of a square, and Fig. 11 for the outside of a contour of a special kind, the loading reducing to a hydrostatic pressure p in all cases. Also shown in Fig. 10 is a solution (dashed line) found by conformal mapping (see G. N. Savin [1]). The cyclic and mirror symmetries have, of course, been taken into account.

Let us consider the integral equations for the bending of simply supported plates. The boundary relations for the functions $\varphi(z)$ and $\psi(z)$ are given in Sec. 16, namely (16.20) and (16.21). Note that a similar boundary value problem arises both in plane stress and in plane strain when a normal component of displacement and a shearing component of stress are prescribed on the boundary.

The first approach to the construction of the integral equations for the bending of simply supported plates has been realized in the work of Z. I. Khalilov [1]. The author proceeds in a direction similar to that which has led N. I. Muskhelishvili to the integral equation of Sec. 18, and obtains a singular integro-differential equation which is then transformed into a Fredholm equation. In the

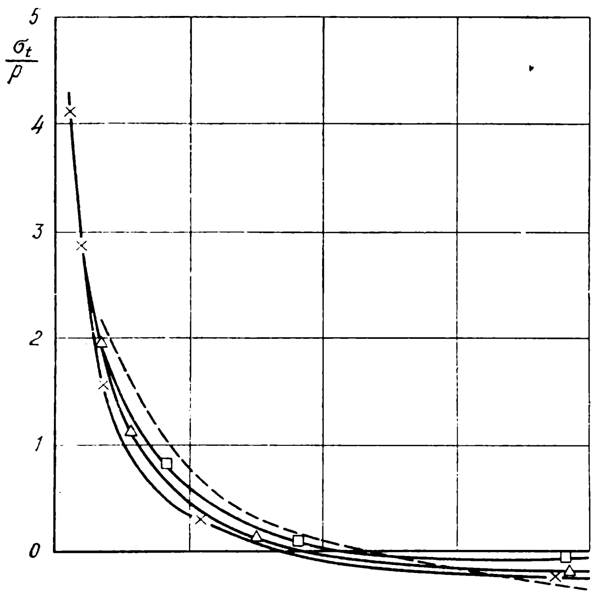

Fig. 10. Distribution of the tangential component of stress in a plate with a square hole under a hydrostatic pressure p (for one octant)

Division of the side of the square: □—10 intervals; △—20 intervals; ×—40 intervals, dashed line—solution of G. N. Savin [1]

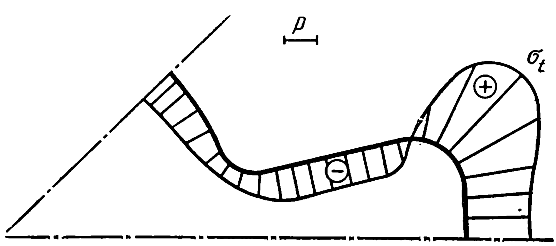

Fig. 11. Distribution of the tangential component of stress in a plate with a complex-shaped cutout under a hydrostatic pressure p (for one octant)

course of the discussion the author had to introduce a restriction on the shape of the contour (the curvature must be different from zero everywhere) dictated only by the techniques used. A procedure suggested by M. M. Fridman [1] is free from this restriction. At the initial stage the author also constructed a singular integro-differential equation, and the investigation was conducted for a multiply connected region.

A. I. Kalandiya [1] used the representations of I. N. Vekua [1] for analytic functions, namely

$$\Phi(z)=\int_L \mu(t)\ln\left(1-\frac{z}{t}\right)ds+\int_L \mu(t)\,ds, \tag{20.6}$$

to obtain a system of singular integral equations and to give a proof of its solvability based on the general principles of the theory of such systems (see N. P. Vekua [1]). It should be noted that at the initial stage the author made an equivalent transformation of the boundary conditions (16.20) and (16.21) by integrating one of them and adding the result to the remaining one. In contrast, D. I. Sherman [18] makes a different transformation of the boundary conditions, namely by differentiation of one of them and addition to the remaining one. The purpose of these transformations is to obtain conditions containing derivatives of the same order. In the work mentioned above the representations constructed for the functions $\varphi(z)$ and $\psi(z)$ led to regular integral equations. The general case of a multiply connected region has been investigated on a similar basis by D. I. Sherman [19].

21. Multiply (Doubly) Connected Regions

Solution of elasticity problems for multiply connected regions has, in fact, already been the subject of the discussion in Secs. 19, 20, where the investigation covers the case of regions of arbitrary connectivity. Moreover, it is possible to solve problems for multiply connected regions by using an alternating algorithm reducing the problem to a set of successively solvable problems for multiply connected regions. For example, for a region bounded from the inside by a contour L_1, and from the outside by a contour L_0, one can proceed as follows: solve the interior problem for the region D_0^+ with a boundary condition prescribed on the contour L_1, next solve the exterior problem for the region D_1^- with a boundary condition representing the difference between the given one and that determined from the solution just obtained, and so on. The questions of the convergence of such a procedure have been discussed by S. L. Sobolev [1]. The actual implementation of this algorithm, however, presents difficulties since when the boundaries are brought closer together the number of such steps must be considerably increased.

Moreover, since the data on the solution of problems by using the Sherman-Lauricella integral equations (or other equations) for multiply connected regions when the distance between the boundaries is small are not available in the literature, of some interest is a special method, though it is actually suitable only for doubly connected regions. The first publication (see D. I. Sherman [12]) was concerned with the solution of a harmonic potential problem (in reference to the torsion of bars) and was subsequently followed by a rather large number of works dealing with the plane problem. A detailed list may be found in the review of D. I. Sherman [21].

This method will be illustrated by considering the plane problem (with stresses given on the contours L_0 and L_1). It is required to determine the functions $\varphi(z)$ and $\psi(z)$ in the region D for the boundary conditions

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f_1(t) \quad (t\in L_0), \tag{21.1}$$

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f_2(t) \quad (t\in L_1). \tag{21.2}$$

We introduce an auxiliary function $\omega(t)$ on one of the contours, say on L_0, defining it by the relation

$$2\omega(t)=\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)} \quad (t\in L_0). \tag{21.3}$$

From conditions (21.1) and (21.3) it follows immediately that

$$\varphi(t)=\omega(t)+0.5f_1(t),$$

$$\psi(t)=-\overline{\omega(t)}-\bar{t}\omega'(t)+0.5[\bar{f}_1(t)+\bar{t}f_1'(t)].$$

It is proved that the functions

$$\varphi_*(z)=\varphi(z)-\frac{1}{2\pi i}\int\limits_{L_0}\frac{\omega(t)+0.5f_1(t)}{t-z}\,dt, \tag{21.4}$$

$$\psi_*(z)=\psi(z)+$$
$$+\frac{1}{2\pi i}\int\limits_{L_0}\frac{\overline{\omega(t)}+\bar{t}\omega'(t)-0.5[\overline{f_1(t)}+\bar{t}f_1'(t)]}{t-z}\,dt \tag{21.5}$$

are analytic in the region D_1^-. We can therefore pass [assuming the function $\omega(t)$ to be given] to an auxiliary problem for this region with suitably modified boundary conditions. We have

$$\varphi_*(t)+t\overline{\varphi_*'(t)}+\overline{\psi_*(t)}=f_2(t)+H(t,\omega,f_1), \tag{21.6}$$

where $H(t, \omega, f_1)$ stands for the sum of terms in the boundary condition formed by the added terms.

Denote by $\varphi_*(z, f_2, H)$ and $\psi_*(z, f_2, H)$ the solution of this boundary value problem (without going into the process of obtaining

it) and, taking up condition (21.3),* express it in terms of the functions $\varphi_*(z)$ and $\psi_*(z)$. The result is a regular Fredholm integral equation, namely

$$\omega(t) = \varphi_*(t_0) - t_0\overline{\varphi_*'(t_0)} - \overline{\psi_*(t_0)} + \\ + \frac{1}{2\pi i}\int_{L_0} \omega(t)\, d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} + \\ + \frac{1}{2\pi i}\int_{L_0} \overline{\omega(t)}\, d\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{4}(f_1' + \bar{f}_1') + \frac{1}{4\pi i}\int_{L_0} f_1(t)\, d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \\ - \frac{1}{4}\frac{1}{\pi i}\int_{L_0} \overline{f_1(t)}\, d\frac{t-t_0}{\bar{t}-\bar{t}_0}. \quad (21.7)$$

It should be noted that in specific problems solved by this method one of the contours was generally a circle (see D.I. Sherman [21]); it is on this contour that the auxiliary function is introduced and expanded at once in a Fourier series, which enables all additional terms to be found without much effort. At the final stage, a system of linear equations is constructed instead of an integral equation. Since this system corresponds to a second-order integral equation, its structure is favourable. In the work of P. I. Perlin [1] it is proved that these systems are quasi-regular under sufficiently general assumptions.

We shall mention the work of P. N. Moshkin [1] where neither of the contours is a circle, they are both taken in the form of ellipses. The author introduces an auxiliary contour, namely a circle in the region occupied by an elastic body, and then solves two independent problems of the equilibrium of a pair of doubly connected regions bounded by a circle, respectively, from the inside and outside, requiring the coincidence of the stresses and displacements on it. It should be noted that the difficulties in calculating the terms defined by the function $\omega(t)$ (if the contour is different from a circle) can also be overcome by using Faber's polynomials.

A periodic problem for a plane with circular holes whose centres lie on the same straight line has been considered by D. I. Sherman [25]. A problem for a plane with three identical circular holes whose centres form an equilateral triangle with a single axis of symmetry in the boundary conditions has been solved by N. M. Kryukova [2]. A problem for a circle with circular holes under conditions of cyclic symmetry has been studied by V. G. Kulish and B. A. Obodovskii [1].

Of course, the foregoing method is also applicable when the boun-

* Recourse to condition (21.1) is not always desirable since a singular integral equation results.

dary conditions prescribed on the contours L_0 and L_1 are not the same. Moreover, it may be assumed that the conditions of the first or second problem are prescribed on one of the contours, say on L_0, and the conditions of the mixed type on the other. As before, it is necessary to introduce a function $\omega(t)$ on the contour L_0 and pass to an auxiliary, now mixed, problem for the region D_1^-. I. G. Aramanovich [1] and I. G. Aramanovich, N. N. Fotieva, V. A. Lytkin [1] implemented this scheme by considering the problem of indenting a half-plane with a circular hole by a punch.

22. Problems of the Theory of Elasticity for Piecewise Homogeneous Bodies

Consider the plane problem of the theory of elasticity for piecewise homogeneous bodies. Let there be a multiply connected region D bounded by smooth contours* L_k $(k = 1, 2, \ldots, m)$ and L_0, all of which, except L_0, lie outside each other, and the contour L_0 encircles all the rest. The region D is filled with an elastic medium having constants $\varkappa$ and μ. The regions D_k^+ bounded by the contours L_k $(k \neq 0)$ are filled with media having constants $\varkappa_k$ and μ_k. Here, for convenience, a constant $\varkappa$ (different in the cases of plane strain and plane stress) has been introduced instead of Lamé's constant λ. The state of stress in each of the regions D_k^+ and D will be described by functions $\varphi_k(z)$, $\psi_k(z)$. Certain conditions are assumed to be prescribed at the junction of the media. The conditions of particular interest in applications are those when the stress vector on L_k varies in a continuous manner and the displacement vector undergoes a given discontinuity $\mu_k(t)$, the function $\mu_k(t)$ belonging to the class H-L. The conditions prescribed on the outer contour may be those of both the first and second (or even mixed) problem.

Below are given all boundary conditions required to determine the functions $\varphi_k(z)$ and $\psi_k(z)$:

$$\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} \quad (t \in L_k), \tag{22.1}$$

$$\varkappa_k\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)} = \\ = \varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + \frac{\mu}{\mu_k}\mu_k(t) \quad (t \in L_k). \tag{22.2}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \quad (t \in L_0). \tag{22.3}$$

For definiteness, the condition of the second fundamental problem has been taken on the contour L_{m+1}.

To begin with, we consider the particular case when all the constants $\varkappa$ and μ are the same. Following D. I. Sherman [5], we rearrange

* The notation is the same as in Sec. 19.

Eqs. (22.1) and (22.2) in the form

$$\varphi_k(t) = \varphi(t) + \frac{1}{1+\varkappa}\mu_k^1(t) \quad (t \in L_k), \tag{22.4}$$

$$\psi_k(t) = \psi(t) + \bar{t}\psi'(t) + \frac{1}{1+\varkappa}[\overline{\mu_k^1(t)} + \bar{t}\mu_k^{1\prime}(t)],$$
$$\mu_k^1(t) = \frac{\mu_0}{\mu_k}\mu_k(t). \tag{22.5}$$

It can be shown that the functions

$$\varphi(z) - \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\mu_k^1(t)}{t-z}\,dt \quad (z \in D),$$

$$\varphi_k(z) - \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\mu_k^1(t)}{t-z}\,dt \quad (z \in D_k^+)$$

represent a single function, analytic in the whole region D_0^+, further denoted by $\varphi_0(z)$. A similar result holds for the functions

$$\psi(z) + \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\overline{\mu_k^1(t)} + \bar{t}\mu_k^{1\prime}(t)}{t-z}\,dt \quad (z \in D),$$

$$\varphi(z) + \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\overline{\mu_k^1(t)} + \bar{t}\mu_k^{1\prime}(t)}{t-z}\,dt \quad (z \in D_k^+);$$

this leads to a function $\psi_0(z)$ analytic in the region D_0^+.

It follows from the preceding discussion that the boundary condition for the functions $\varphi_0(z)$ and $\psi_0(z)$ may be written as

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f(t) + H(t, \mu_k^1) \quad (t \in L_0), \tag{22.6}$$

where $H(t, \mu_k^1)$ stands for the terms defined by the functions $\mu_k^1(t)$. The calculation of all additional terms entering into the expressions for the functions $\varphi_k(z)$ and $\psi_k(z)$ is performed in an elementary way in the case when the contours L_k are circles (by expanding the functions in Fourier series). In other cases it is advisable to resort to Faber's polynomials (see, for example, Yu. A. Amenzade, T. Yu. Agaev [1] and Yu. N. Shalyukhin [1]).

Note that the foregoing approach can also be used in cases where some of the regions D_k^+ themselves have holes filled with elastic media when conditions of the type of (22.1) and (22.2) are fulfilled on the contours.

We now turn to the consideration of the general case when the elastic constants are distinct. We shall describe a method proposed by S. G. Mikhlin [1] where the complex Green's function for each

of the regions D_k^+ is assumed to be known. This makes it possible to express the functions $\varphi_k(z)$ and $\psi_k(z)$ explicitly (by means of a Cauchy integral in terms of regular terms) directly from the boundary conditions (22.1), or (22.2), assuming both sides of the equality to be known. After this, by using equality (22.2) or (22.1), respectively, the author arrives at an always solvable system of singular integral equations of the $(m+1)$th order.

In the work of D. I. Sherman [9], representations of the type of (19.2) and (19.3) are prescribed for the functions $\varphi_k(z)$ and $\psi_k(z)$ in each of the regions D_k^+ and then a limiting process is applied to pass to the points of the contours L_k and L_0. Equalities (22.1) to (22.3) then lead to regular integral equations of order $2m+1$. Representations of this kind have been used by L. A. Fil'shtinskii [2] to construct regular integral equations in the case of a doubly periodic problem when a parallelogram of periods contains a number of inclusions.

Let us now discuss in greater detail the method of D. I. Sherman [20] reducing the problem to a system of singular equations of the second kind of order $m+1$. For simplicity, we restrict ourselves to the case when $m=1$. The following relations hold on the contour L_1:

$$\begin{gathered}\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}=\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)},\\ \varkappa_1\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}=\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}+\frac{\mu}{\mu_1}\mu_1(t).\end{gathered}\tag{22.7}$$

We introduce an auxiliary function $\omega(t)$ according to the equality

$$\omega(t)=\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}-\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\quad(t\in L_1).\tag{22.8}$$

Construct Cauchy-type integrals:

$$\varphi^*(z)=\frac{1}{(1+\varkappa_1)}\frac{1}{2\pi i}\int_{L_1}\frac{\omega(t)}{t-z}\,dt,$$

$$\psi^*(z)=\frac{1}{(1+\varkappa_1)}\cdot\frac{1}{2\pi i}\int_{L_1}\frac{\overline{\omega(t)}+\bar{t}\omega'(t)}{t-z}\,dt.$$

By means of the equalities

$$\begin{gathered}\varphi_0(z)=\varphi(z)-\varphi^*(z),\quad \psi_0(z)=\psi(z)+\psi^*(z)\quad(z\in D),\\ \varphi_0(z)=\varphi_1(z)-\varphi^*(z),\quad \psi_0(z)=\psi_1(z)+\psi^*(z)\quad(z\in D_1^+)\end{gathered}$$

we can then pass to the problem for a solid region: $\varphi_0(t)+t\overline{\varphi_0'(t)}+\overline{\psi_0(t)}=H(\omega,t)$.

The solution of this problem is represented in symbolic form as

$$\varphi_0(z)=H_1(\omega,z),\ \psi_0(z)=H_2(\omega,z).\tag{22.9}$$

We now make an inverse transformation to the functions $\varphi_k(z)$ and $\psi_k(z)$ and use the second relation of (22.7); it leads to a singular integral equation, namely

$$a\omega(t_0)+\frac{b}{\pi i}\int_{L_1}\frac{\omega(t)}{t-t_0}\,dt+N(\omega,\ t_0)=\frac{\mu}{\mu_1}\mu_1(t),$$
$$a=\frac{1}{2}\left[\frac{\varkappa_1+1}{\mu_1}+\frac{\varkappa+1}{\mu}\right],\quad b=\frac{1}{2}\left[\frac{\varkappa_1-1}{\mu_1}-\frac{\varkappa-1}{\mu}\right]. \tag{22.10}$$

The terms appearing on the left-hand side of Eq. (22.10) and denoted all together by $N(\omega,\ t_0)$ are regular operators in the unknown function $\omega(t)$. The foregoing method has found wide application (see Yu. A. Amenzade [1]).

It should be noted that the above methods enable one to obtain integral equations for a more general case when there are a number of other boundary contours inside some contours L_k. These and related topics are studied in Sec. 36 from the general standpoint of the three-dimensional problem.

Chapter V

SOME SPECIAL TOPICS OF TWO-DIMENSIONAL ELASTICITY

23. Problems of the Theory of Elasticity for Bodies with Cuts

Let there be m smooth unclosed non-intersecting contours L_k in the plane of the complex variable z and let their ends be denoted by a_k and b_k, respectively. It is required to solve the elasticity problem for a plane with cuts along the contours L_k assuming stresses or displacements to be given on the sides of the cuts. For definiteness, the following study will be limited to the second fundamental problem. It is necessary to determine the functions $\varphi(z)$ and $\psi(z)$ in the whole plane under the conditions

$$\varphi^+(t) + t\overline{\varphi^{+\prime}(t)} + \overline{\varphi^+(t)} = f^+(t) + C_k \tag{23.1}$$

$$(t \in L_k).$$

$$\varphi^-(t) + \overline{t\varphi^{-\prime}(t)} + \overline{\varphi^-(t)} = f^-(t) + C_k \tag{23.2}$$

The plus sign indicates that the limit is taken on the left when moving from the point a_k to the point b_k and the minus sign, on the right (as in Sec. 2). The functions $f^\pm(t)$ are assumed to be given and belonging to the class H-L. The constants C_k are to be found in the course of the solution. The condition that the resultant vector of the external forces is zero accounts for the single-valuedness of the functions $f^\pm(t)$.

The solutions of problems of this kind for special cases when all cuts are situated in the same straight line or circle are effectively found by the method of linear relationship (Secs. 26, 27).

Below is given the derivation of the general singular integral equation as a further development of the method described in Sec. 22 when considering problems for composite bodies.

We first simplify the formulation of the problem assuming that the functions $f^+(t)$ and $f^-(t)$ coincide (in other words, assuming that the stress vector varies in a continuous manner across a cut).

It is not difficult to pass from the general case of loading to the one indicated. To do this, it is necessary to introduce additional functions in the whole plane, namely

$$\psi_0(z)=\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{f^-(t)-f^+(t)}{t-z}\,dt,\quad \varphi_0(z)=0.$$

For the new functions $\varphi_*(z)=\varphi(z)$ and $\psi_*(z)=\psi(z)-\psi_0(z)$ we then arrive at the required conditions

$$\varphi_*^+(t)+t\,\overline{[\varphi_*^+(t)]'}+\overline{\psi_*^+(t)}=\varphi_*^-(t)+t\,\overline{[\varphi_*^-(t)]'}+\overline{\psi_*^-(t)}=$$
$$=\frac{f^+(t)+f^-(t)+2C_k}{2}=f(t)+C_k. \quad (23.3)$$

We introduce auxiliary functions $\mu_k(t)$ on the cuts L_k defined by the following representations (the subscript $*$ is dropped):

$$\mu_k(t)=\varkappa\varphi^+(t)-t\overline{\varphi^{+\prime}(t)}-\overline{\psi^+(t)}-\varkappa\varphi^-(t)+t\overline{\varphi^{-\prime}(t)}-\overline{\psi^-(t)}. \quad (23.4)$$

The physical meaning of the functions $\mu_k'(t)$ is obvious, they represent a jump in the displacement (apart from the factor $1/2\,\mu$). It is therefore natural to include in the formulation of the problem the condition that these functions vanish at the ends of the segments.

By direct substitution we verify that the functions

$$\varphi(z)=\sum_{k=1}^{m}\frac{1}{2\pi i(1+\varkappa)}\int_{L_k}\frac{\mu_k(t)}{t-z}\,dt,$$

$$\psi(z)=-\sum_{k=1}^{m}\frac{1}{2\pi i(1+\varkappa)}\int_{L_k}\frac{\overline{\mu_k(t)}+\bar t\mu_k'(t)}{t-z}\,dt$$

identically satisfy the first of the conditions contained in (23.3), which expresses the continuity of the external stresses on the cuts, If we refer to the second of equalities (23.3) defining these stresses, we arrive by using formulas (2.9) at a singular integral equation. namely

$$\sum_{k=1}^{m}\left\{\frac{1}{\pi i}\int_{L_k}\frac{\mu_k(t)}{t-t_0}\,dt+\frac{1}{2\pi i}\int_{L_k}\mu_k(t)\,d\ln\frac{\bar t-\bar t_0}{t-t_0}-\right.$$
$$\left.-\frac{1}{2\pi i}\int_{L_k}\overline{\mu_k(t)}\,d\frac{\bar t-\bar t_0}{t-t_0}\right\}=(\varkappa+1)\,[f(t_0)+C_k]\quad (t_0\in L_k). \quad (23.5)$$

Since the solution of this equation should be sought in the class of functions bounded at the end points (it can be shown, by using

the expression for a canonical function, that the condition for boundedness implies that the solution is zero, as assumed in the formulation of the problem), the index of the equation is found to be $-m$. The equation is therefore solvable only if condition (18.6) is fulfilled, the condition expressing the orthogonality of the real component of the right-hand side to all eigenfunctions of the companion equation. If the cuts are situated in the same straight line (along the real axis), the equation becomes simpler and takes an elementary form:

$$\sum_{k=1}^{m} \frac{1}{\pi i} \int_{L_k} \frac{\mu_k(t)}{t-t_0}\, dt = (\varkappa+1)\,[f(t_0)+C_k] \quad (t_0 \in L_k). \tag{23.6}$$

In a number of works (see, for example, L. L. Libatskii [1], L. L. Libatskii, S. T. Baranovich [1] and L. L. Libatskii, M. I. Bida [1]) the authors use, instead of Eq. (23.6), an equation obtained from it by differentiation with respect to the variable t_0 followed by interchanging the order of integration and differentiation, which leads to a singular integral equation in the functions $\mu_k'(t)$:

$$\sum_{k=1}^{m} \frac{1}{\pi i} \int_{L_k} \frac{\mu_k'(t)}{t-t_0}\, dt = (\varkappa+1)\, f'(t_0) \quad (t_0 \in L_k). \tag{23.7}$$

Since in this case we are seeking an unbounded solution at all ends, it will contain arbitrary constants, which can be determined from the condition for single-valuedness of displacements when passing round the contours L_k.

Unfortunately, a comparative analysis of the above approaches is not available in the literature. We shall only note that in the works of L. L. Libatskii *et al.* mentioned above and also in the work of V. V. Kopasenko it is recommended that the solution $\mu_k'(t)$ should be represented as the product of a series containing undetermined coefficients and an appropriate factor incorporating the singularity of the solution; the solution may also be represented as a sum consisting of terms of a series and an additional term (in the form of the same factor) with coefficients that are to be determined simultaneously.

We now turn to the consideration of the case when the region occupied by an elastic body is bounded (from the outside or inside) by a contour L. The functions $\varphi(z)$ and $\psi(z)$ are sought in the form of sums:

$$\varphi(z) = \varphi_0(z) + \sum_{k=1}^{m} \frac{1}{(1+\varkappa)\,2\pi i} \int_{L_k} \frac{\mu_k(t)}{t-z}\, dt, \tag{23.8}$$

$$\psi(z) = \psi_0(z) - \sum_{k=1}^{m} \frac{1}{(1+\varkappa)\,2\pi i} \int_{L_k} \frac{\overline{\mu_k(t)} + \bar{t}\mu_k'(t)}{t-z}\, dt, \tag{23.9}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are analytic functions in a solid region D. We arrive at a system of singular equations differing from (23.7) only by the term $[\varphi_0(t_0) + \overline{t_0\varphi_0'(t_0)} + \overline{\psi_0(t_0)}]$. In addition, we must introduce equations taking account of the boundary conditions on the contour L. By using any constructive representation of the functions $\mu_k(t)$, we can pass to the determination of the functions $\varphi_0(z)$ and $\psi_0(z)$ (with the assumed right-hand side) and, by inverse transformation to the functions $\varphi(z)$ and $\psi(z)$, determine the preassigned parameters in the representation of the functions $\mu_k(t)$ from the boundary conditions on the contours L_k.

It is natural to expect that in the case of a comparatively close spacing between the cuts and also between the cuts and the boundary contour the actual construction of the solution involves some complication, and to overcome this we must resort to more refined calculations (see S. Ya. Yarema [1]). When the cuts are spaced sufficiently far apart, the alternating method is believed to be efficient (see A. M. Lin'kov [1]).

The foregoing method of solving problems for bodies* with cuts consists in reducing them to a problem for a solid region (by certain additions). We may proceed in a different way, namely by considering an auxiliary problem for the whole plane (with the same cuts) (see S. S. Zargaryan, R. L. Enfiadzhyan [1]). To do this, we must introduce an unknown function on the outer contour just as the function $\omega(t)$ was introduced in considering doubly connected regions (Sec. 21). The specific form assigned to this function depends on the nature of the boundary conditions. After constructing the appropriate additional terms (in the form of Cauchy-type integrals), we transform to a problem for the whole plane. By solving the latter in some way, we obtain a Fredholm integral equation of the second kind for the initially introduced function.

The calculation data given in the work mentioned above relate to the case when the cut is far removed from the outer boundary, and this prevents a very desirable comparison of the two approaches discussed above.

It is natural to expect that as the distance between the cut and the outer boundary is decreased, the convergence of the above algorithms becomes worse as is the case in solving problems for doubly connected, and in general multiply connected, regions. We refer to the work of Yang Wei Hguin [1] where this statement is made in solving a similar singular equation, though arising in studying a different problem, namely that of bending a plate with an interior rigid support.

Note that a number of authors (see R. J. Hartranft, G. C. Sih [2]

* Other than a complete plane.

apply the alternating method to solve problems of this kind, a method consisting in successively considering problems for a body with an internal cut and for a solid one.

Consider problems of the bending of plates with cuts (see A. M. Lin'kov, V. A. Merkulov [1]). Before proceeding to mathematical formulation, we note that the statement of such problems (in their pure form) is at variance with the principles underlying the bending theory since problems of this kind are essentially three-dimensional. Nevertheless, they have a definite practical significance.

In this case (when the bending moments and the shearing forces are zero) we arrive at boundary conditions on the cuts L_k similar to (23.1), (23.2), namely

$$\varkappa\varphi^+(t) - t\overline{\varphi^{+\prime}(t)} - \overline{\psi^+(t)} = f^+(t) + iD_k t + C_k \quad (t \in L_k), \quad (23.10)$$

$$\varkappa\varphi^-(t) - t\overline{\varphi^{-\prime}(t)} - \overline{\psi^-(t)} = f^-(t) + iD_k t + C_k. \quad (23.11)$$

The functions $f^+(t)$ and $f^-(t)$ stand for the expressions determined by a particular solution of the non-homogeneous bending equation (see Sec. 18), D_k are real constants, C_k are, as before, complex constants. Thus, the only difference from the plane problem is that the right-hand side involves the functions $D_k(t)$.

By introducing an auxiliary function [similar to (23.4)] a singular integral equation is obtained. Here, too, the auxiliary function must vanish at the ends. Moreover, we must take into account the conditions for single-valuedness of displacements (when passing round each cut). All these conditions lead to a system of equations for the constants C_k and D_k, and the solvability of this system follows from the uniqueness of the solution of the bending problem itself.

24. Integral Equations for Mixed (Contact) Problems

We now turn to the formulation of a mixed problem of the theory of elasticity. Let a body occupy a finite region D^+ bounded by a smooth contour L. Let non-overlapping arcs L_k' be chosen on this contour with their ends at points a_k, b_k ($k = 1, 2, \ldots, m$). As before, the sense of description is such that the region under consideration is on the left. The remaining portions of the contour are denoted by L_k'' (b_k, a_{k+1})*; we thus obtain systems of arcs L' and L''.

The mixed problem of the theory of elasticity consists in determining the functions $\varphi(z)$ and $\psi(z)$ in the region D^+ from the boundary

* By a_{m+1} is meant the point a_1.

conditions

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t)+C_k \quad (t\in L'), \tag{24.1}$$

$$-\varkappa\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t) \quad (t\in L''), \tag{24.2}$$

$$f(t)=i\int_{a_k}^{t}(X_n+iY_n)\,ds \quad (t\in L'),$$

$$f(t)=2\mu(u+iv) \quad (t\in L'').$$

Here C_k is a constant on each arc L' to be determined in the course of the solution of the problem. The function $f(t)$ belongs to the class H-L on all arcs of the contour L.

To construct an integral equation, we use the same representations for the functions $\varphi(z)$ and $\psi(z)$ as in Sec. 19, viz. (19.2) and (19.3). On the basis of formulas (2.9) we obtain the limiting values (from the inside) for the functions $\varphi^+(t)$, $\varphi^{+\prime}(t)$, and $\psi^+(t)$. Substituting these values in the boundary conditions (24.1), we arrive at the required equation

$$K\omega\equiv A(t_0)\,\omega(t_0)+\frac{B(t_0)}{\pi i}\int_L\frac{\omega(t)}{t-t_0}\,dt+\int_L k_1(t_0,t)\,\omega(t)\,dt+$$
$$+\int_L\overline{k_2(t_0,t)}\;\overline{\omega(t)}\;\overline{dt}=f(t_0)+C(t_0). \tag{24.3}$$

Here

$$k_1(t_0,t)=\frac{\varkappa}{2\pi i}\frac{\partial}{\partial t}\ln\frac{\bar t-\bar t_0}{t-t_0},\qquad k_2(t_0,t)=\frac{1}{2\pi i}\frac{\partial}{\partial t}\frac{\bar t-\bar t_0}{t-t_0},$$

$$\{A(t_0),\ B(t_0),\ C(t_0)\}=\begin{cases}(1-\varkappa)/2, & (1+\varkappa)/2, & C_k\\ -\varkappa & 0, & 0\end{cases}\quad\begin{matrix}(t_0\in L')\\ (t_0\in L'').\end{matrix}$$

The constants C_k are determined from the condition for the solvability of the integral equation in the class h_{2m}, which leads to the system

$$\operatorname{Re}\int_L[f(t)+C(t)]\,\sigma_k(t)\,dt=0 \quad (k=1,\ 2,\ \ldots,\ m), \tag{24.4}$$

where $\sigma_k(t)$ is the complete system of solutions of the class h_0 for the companion equation.

The investigation of the solvability of Eq. (24.3), and in particular of system (24.4), has been carried out by G. F. Mandzhavidze [1].

We suggest a procedure whose implementation does not require finding the functions $\sigma_k(t)$. We seek a solution bounded at any m ends. The equation is then always solvable (since the index is zero), the solution depending on the constants C_k. In addition, we now require that the solution should be bounded at the remaining ends. We then arrive at a system of equations equivalent to system (24.4) and hence necessarily solvable.

Note that regular equations have been obtained for the mixed problem when either Schwarz' function is known (S. G. Mikhlin [4]) or the function effecting the conformal mapping is known (D. I. Sherman [3]). In the latter case the solution of the equation is constructed explicitly if the mapping function is rational.

In conclusion, we turn to the consideration of mixed problems arising in the theory of plate bending. Naturally, the case when the plate is clamped in some portions of the boundary and free in the remaining portions falls into the class of mixed plane problems discussed previously [see conditions (15.14) and (15.15)]. We shall only consider, therefore, problems where simply supported-edge conditions are fulfilled in some portions of the contour while in the remaining portions there may be either clamped-edge or free-edge conditions (or a combination of both).

A. I. Kalandiya [1] has studied the most general problem of plate bending with boundary conditions of mixed type (a combination of simply supported, clamped, and free edges). Initially the author transforms the boundary conditions and, after introducing representations for the functions $\varphi(z)$ and $\psi(z)$ in the form of (19.2) and (19.3), arrives at a system of singular integral equations for the real and imaginary parts of the auxiliary function. Further the author investigates this system (on the basis of the general theory of N. P. Vekua [1]), proves its solvability, and establishes the behaviour of the solution at the points of transition of one boundary condition to another.

In reference to the case when there are no portions where the plate is free the author simplifies the results by transforming the derived system of singular equations into a regular one.

We shall mention two works of D. I. Sherman [16, 17] dealing with problems of the bending of a circular plate. In the first work it is assumed that the plate is clamped on one semicircle, and simply supported on the other. In the second work the plate is free on one semicircle, and simply supported on the other. In both cases the author directly obtains singular integral equations which are solved by expanding the unknown function in a series. The resulting systems of algebraic equations are completely regular. If the arcs are other than a semicircle, it is advisable to make a change of the variables in the integral equation by a linear fractional transformation.

25. Problems of the Theory of Elasticity for Bodies Bounded by Piecewise Smooth Contours

To solve elasticity problems in the case when the boundary contour is piecewise smooth, we cannot, obviously, directly apply the regular integral equations given in Chap. IV (or singular equations that will come under consideration in Chap. VI). True, the formal use of the corresponding integral representations is possible, but the kernels of the equations are discontinuous, and moreover the coefficient of the term outside the integral sign takes a different value at an angular point than at smooth points.

Note that in the case of one angular point it is possible to construct an integral equation of a special class (see S. M. Belonosov [1]). The author conformally maps a given region onto a half-plane and, applying the Laplace transformation, arrives at an integral equation* with a Carleman-type kernel. The solvability of this equation is proved and considerations are stated concerning an efficient numerical implementation.

Comparison of the integral equations for the original region and a region close to it (with rounded corners) permitted the author to prove that the solution of the boundary value problems are close everywhere except in a small neighbourhood of an angular point.

It should be noted that an asymptotic expansion of the solution of general elliptic boundary value problems in terms of a parameter characterizing the rounding has been constructed in the work of V. G. Maz'ya, S. A Nazarov and B. A. Plamenevskii [1]. The implementation of these results in reference to the plane problem of the theory of elasticity has been accomplished by S. A. Nazarov [1].

The foregoing provides the justification for using the solutions of problems for regions with smoothed boundaries to determine the stresses and displacements outside the neighbourhood of angular points in the case of irregular boundaries. Naturally, these solutions can be obtained on the basis of the Muskhelishvili and Sherman-Lauricella integral equations. As to the asymptotic behaviour of the solution at angular points, it should be noted that the factors multiplying the eigenfunctions (see Sec. 7) (actually determining the asymptotic behaviour) can also be found with sufficient accuracy from approximate solutions.

Some considerations for the general case of a three-dimensional problem will be given in Secs. 35 and 37. Below is described a procedure suggested by M. Stern [1]. For definiteness, it is assumed that the region contains an unloaded cut intersecting the outer boundary (Fig. 12). Let origin of coordinates be taken as the end of the cut. According to formulas (17.4) with $\alpha = \pi$, the displacements and

* In the case of smooth contours the equations are Fredholm ones.

stresses in the neighbourhood of an angular point may be represented as

$$
\begin{aligned}
U_r &= \frac{1}{4\mu}\left(\frac{r}{2\pi}\right)^{1/2}\left\{\left[(2k-1)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]K_{\rm I}-\right.\\
&\qquad\left.-\left[(2k-1)\sin\frac{\theta}{2}-3\sin\frac{3\theta}{2}\right]K_{\rm II}\right\}+o(r^{1/2}),\\
U_\theta &= \frac{1}{4\mu}\left(\frac{r}{2\pi}\right)^{1/2}\left\{\left[-(2k+1)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]K_{\rm I}-\right.\\
&\qquad\left.-\left[(2k+1)\cos\frac{\theta}{2}-3\cos\frac{3\theta}{2}\right]K_{\rm II}\right\}+o(r^{1/2}),\\
\sigma_r &= \frac{1}{4(2\pi r)^{1/2}}\left\{\left(5\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right)K_{\rm I}-\right.\\
&\qquad\left.-\left(5\sin\frac{\theta}{2}-3\sin\frac{3\theta}{2}\right)K_{\rm II}\right\}+o(r^{1/2}),\\
\sigma_\theta &= \frac{1}{4(2\pi r)^{1/2}}\left\{\left(3\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\right)K_{\rm I}-\right.\\
&\qquad\left.-3\left(\sin\frac{\theta}{2}+3\sin\frac{3\theta}{2}\right)K_{\rm II}\right\}+o(r^{1/2}),\\
\tau_{r\theta} &= \frac{1}{4(2\pi r)^{1/2}}\left\{\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)K_{\rm I}+\left(\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right)K_{\rm II}\right\}+\\
&\qquad+o(r^{1/2}),
\end{aligned}
\tag{25.1}
$$

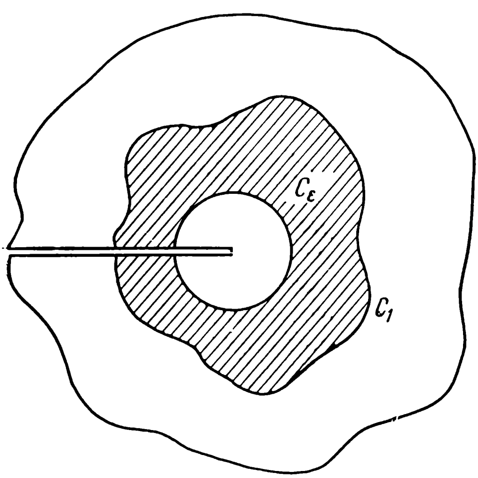

Fig. 12. Auxiliary contours in the region with a cut

where $k = 3 - 4\nu$ in plane strain and $k = (3-\nu)/(1-\nu)$ in plane stress. Here some rearrangement of terms has been made in comparison with (8.26) to represent the results in the form used in fracture

mechanics (see V. Z. Parton, E. M. Morozov [1]). Indeed, we have

$$K_{\mathrm{I}} = \lim_{r\to 0} (2\pi r)^{1/2}\sigma_\theta|_{\theta=0}, \quad K_{\mathrm{II}} = \lim_{r\to 0} (2\pi r)^{1/2}\tau_{r\theta}|_{\theta=0}. \quad (25.2)$$

We now introduce into consideration a displacement field $\hat{u}$ and its associated stress field:

$$\begin{aligned}
\hat{U}_r &= \frac{1}{2(2\pi r)^{1/2}(1+k)}\Big\{\Big[(2k+1)\cos\frac{3\theta}{2} - 3\cos\frac{\theta}{2}\Big]C_1 + \\
&\qquad + \Big[(2k+1)\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\Big]C_2\Big\},\\
\hat{U}_\theta &= \frac{1}{2(2\pi r)^{1/2}(1+k)}\Big\{\Big[-(2k-1)\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2}\Big]C_1 + \\
&\qquad + \Big[(2k+1)\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}\Big]C_2\Big\},\\
\hat{\sigma}_r &= -\frac{\mu}{2(2\pi r^3)^{1/2}(1+k)}\Big\{\Big[7\cos\frac{3\theta}{2} - 3\cos\frac{\theta}{2}\Big]C_1 + \\
&\qquad + \Big[7\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\Big]C_2\Big\},\\
\hat{\sigma}_\theta &= -\frac{\mu}{2(2\pi r^3)^{1/2}(1+k)}\Big\{\Big[\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\Big]C_1 + \\
&\qquad + \Big[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\Big]C_2\Big\},\\
\hat{\tau}_{r\theta} &= -\frac{\mu}{2(2\pi r^3)^{1/2}(1+k)}\Big\{3\Big[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\Big]C_1 - \\
&\qquad - \Big[\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\Big]C_2\Big\},
\end{aligned} \quad (25.3)$$

where C_1 and C_2 are arbitrary constants.

We now draw a sufficiently arbitrary contour C in the body joining the sides of the cut and sufficiently far removed from the origin. We apply Betti's formula (14.15) to the solution sought and to solution (25.3) introduced above. Let a contour of integration be chosen consisting of the arc C_1, the corresponding sides of the cut, and a circle of small radius C_ε. Note that the solution sought has the asymptotic representation (25.1) on the arc C_ε. In Betti's formula the integrals over the line segment vanish, and the integral over the arc C_ε is evaluated in closed form as $\varepsilon \to 0$. It can be shown that the following equality holds:

$$\begin{aligned}
I_\varepsilon = -\int_{C_\varepsilon} (UT_\nu\hat{U} - \hat{U}T_\nu U)\,ds = \int_{-\pi}^{\pi} (\hat{\sigma}_r U_r - \hat{\tau}_{r\theta}U_\theta - \sigma_r\hat{U}_r + \tau_{r\theta}\hat{U}_\theta)\times r\,d\theta = \\
= \int_C (UT_\nu\hat{U} - \hat{U}T_\nu U)\,ds = c_1K_{\mathrm{I}} + c_2K_{\mathrm{II}}. \quad (25.4)
\end{aligned}$$

Thus, if there is a solution of the problem sufficiently valid outside the neighbourhood of an angular point, by referring twice to formula (25.4), and setting $c_1 = 1$ and $c_2 = 0$, and vice versa $c_1 = 0$ and $c_2 = 1$, we arrive at values for the factors K_{I} and K_{II}.

Note that the calculation data of M. Stern, E. B. Becker, R. S. Dunham [1] demonstrate the efficiency of this procedure in the case of the mixed problem; the approximate solutions have been constructed by the finite element method. Of course, the algorithm can be extended, with some error, to the case when curved portions of the boundary are adjacent to an angular point.

V. G. Maz'ya and B. A. Plamenevskii [1] have proposed a method for determining the coefficients of eigenfunctions in the case of boundary value problems for elliptic systems. Its implementation requires the construction of the solution of an auxiliary boundary value problem for the same region and the adjoint equation; the solution must have known singularities at angular points. The resulting representation for the unknown coefficients takes the form of a boundary integral of an expression involving the given boundary conditions and the solution mentioned above.

In reference to elasticity problems the algorithm is simplified since the auxiliary problem is a problem for the same equations. The required formula follows from Betti's identity applied to the solution sought and the solution having the asymptotic representations (25.3).

Note that N. F. Morozov [1] gives examples illustrating the above method.

Let us consider an approximate approach to the solution of problems for regions of the class under consideration on the basis of the numerical implementation of the integral equations obtained in Secs. 18 and 19. (This question is studied in detail from the general standpoint in Sec. 37.) The point is that the use of the same approaches as in the case of smooth contours leads to integral equations differing only at angular points. By suitably choosing the pivotal points (they must not coincide with the angular points), it is therefore possible to evade the question of constructing the equations themselves and to extend formally* each of the algorithms of numerical solution for the case of smooth contours to piecewise smooth ones.

We shall mention the work of Yu. N. Shalyukhin [1] in which the author proposes a method for solving Muskhelishvili's equation (as well as the Sherman-Lauricella equation) specially adapted to the case of polygonal contours. The unknown function on each of the segments is given in the form of a polynomial with coefficients that are to be determined. It is possible to obtain explicit expressions for all integral terms of the equation. The above coefficients

* Without rigorous mathematical proof.

can be determined, for example, from the condition for the coincidence of the left-hand and right-hand sides of the equation at the corresponding number of interior points of the contour.

26. Method of Linear Relationship

The foregoing results show that practically in all cases the fundamental problems for bodies with cuts and also mixed problems can be reduced to singular integral equations. The solution of the latter is sometimes carried out by resorting to the Riemann boundary value problem.

A convenient method is that of reducing problems of the classes under consideration (in some special cases) directly to the Riemann boundary value problem. This method is known in the literature as the *method of linear relationship.*

Let D^+ be the upper half-plane, and D^- the lower one. Consider an analytic function $\Phi(z)$ in the region D^+ and define a function $\overline{\Phi}(z)$ in the region $\overline{D^-}$:

$$\overline{\Phi}(z) = \overline{\Phi(\overline{z})} \quad (z \in D^+). \tag{26.1}$$

Below are given some relations, which will be needed in what follows, between the limiting values of the functions $\Phi(z)$ and $\overline{\Phi}(z)$ $(z \to t,\ z \in D^+,\ \overline{z} \in D^-)$:

$$\overline{\Phi}^-(t) = \overline{\Phi^+(t)}, \quad \overline{\Phi^-(t)} = \Phi^+(t). \tag{26.2}$$

From these relations it follows, in particular, that if $\operatorname{Im} \Phi^+(t) = 0$ in some portion of the real axis, the continuation of the function by linear relationship is analytic.

Consider the elasticity problem for the half-plane D^-. Suppose that the following representations hold for functions $\Phi(z)$ and $\Psi(z)$:*

$$\Phi(z) = -\frac{X+iY}{2\pi}\frac{1}{z} + O\left(\frac{1}{z^2}\right) \qquad \Psi(z) = \frac{X-iY}{2\pi}\frac{1}{z} + O\left(\frac{1}{z^2}\right), \tag{26.3}$$

where X and Y are the projections of the resultant vector on the x and y axes. Let us continue the functions $\Phi(z)$ and $\Psi(z)$ into the upper half-plane D^+ by applying the method of linear relationship, i.e., by introducing into consideration the functions $\overline{\Phi}(z)$ and $\overline{\Psi}(z)$. We define a function $\Phi_1(z)$ in the region D^+:

$$\Phi_1(z) = -\overline{\Phi}(z) - z\overline{\Phi}'(z) - \overline{\Psi}(z) \quad (z \in D^+). \tag{26.4}$$

* In conformity with the results of Sec. 15 and the restrictions imposed on the behaviour of stresses at infinity.

Substituting the point $\bar{z}$ ($\bar{z} \in D^-$) in this formula, and taking the conjugate of both sides, we arrive, by (26.1), at a new representation:

$$\bar{\Phi}_1(z) = -\Phi(z) - z\Phi'(z) - \Psi(z). \tag{26.5}$$

Relation (26.5) enables one to express the function $\Psi(z)$ in terms of two functions, $\Phi(z)$ and $\bar{\Phi}_1(z)$. The first function is analytic in the region D^-, and the second in the region D^+. Thus, the state of stress and strain in the region D^- can be represented in terms of two functions (with different regions of analiticity).

Below is given the expression for the combination of stresses $\sigma_y - i\tau_{xy}$, which follows from formulas (15.9) if the function $\Psi(z)$ is replaced with $\Phi(z)$ and $\Phi_1(z)$ according to (26.5):

$$\sigma_y - i\tau_{xy} = \Phi(z) + (z - \bar{z})\,\overline{\Phi'(z)} - \Phi_1(\bar{z}). \tag{26.6}$$

Suppose that the function $\Phi(z)$ has the limiting values $\Phi^-(t)$ at almost all points of the real axis, with the exception of some points where nevertheless the following equality holds: $\lim\limits_{z\to t} y\overline{\Phi'(z)} = 0$. By applying a limiting process in formula (26.6), we pass to the points of the real axis; omitting the subscript on the function $\Phi_1(z)$ (there is no need for it now),* we arrive at the Riemann boundary value problem:

$$\Phi^-(t) - \Phi^+(t) = \sigma_y - i\tau_{xy} = f(t), \tag{26.7}$$

where $f(t)$ is a given function

Thus, the solution of the second fundamental problem for a half-plane [if (26.3) is fulfilled] can at once be represented in terms of a Cauchy-type integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z}\, dt \tag{26.8}$$

We now turn to the consideration of the first fundamental problem. Differentiate the displacement boundary condition (15.15) in the direction of the x axis:

$$\varkappa\Phi(z) - \overline{\Phi(z)} - z\overline{\Phi'(z)} - \overline{\Psi(z)} = 2\mu(u' + iv') = f(t). \tag{26.9}$$

Further, referring again to representation (26.5), we obtain

$$2\mu(u' + iv') = \varkappa\Phi(z) + (\bar{z} - z)\,\overline{\Phi'(z)} + \Phi_1(\bar{z}). \tag{26.10}$$

---

* The functions $\Phi(z)$ and $\Phi_1(z)$ will be further regarded as the corresponding values in D^- and D^+ of a single piecewise analytic function denoted by $\Phi(z)$.

On passing to the points of the real axis by a limiting process, we arrive at the Riemann boundary value problem:

$$\varkappa\Phi^-(t)+\Phi^+(t)=f(t). \tag{26.11}$$

The solution of the boundary value problem (26.11) can be found in explicit form. To do this, we must pass to an auxiliary piecewise analytic function $\Omega(z)$ as follows:

$$\Omega(z)=\Phi(z)\quad(y>0),\quad \Omega(z)=-\varkappa\Phi(z)\quad(y<0).$$

The function $\Omega(z)$ is then found from the boundary value problem

$$\Omega^+(t)-\Omega^-(t)=f(t),$$

whose solution is also represented in the form of (26.8).

A combination of conditions (26.7) and (26.11) enables one to obtain a boundary value problem for the case of the mixed problem of the theory of elasticity. Naturally, in this case the coefficients are discontinuous. The questions of application of this method to the solution of contact problems are discussed in the work of N. I. Muskhelishvili [3], where the author studies various cases of boundary conditions (smooth punches, friction between a punch and a deformable body), and also the case of the simultaneous deformation of two elastic half-planes with different coefficients $\varkappa$ and μ.

We now turn to the elasticity problem for a plane with straight cuts $L=L_1+L_2+\ldots+L_m$ situated along one straight line taken as the real axis. The functions $\Phi(z)$ and $\Psi(z)$, analytic in the whole plane, with the exception of the cuts, are of the form

$$\begin{gathered}\Phi(z)=-\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z}+O\left(\frac{1}{z^2}\right),\\ \Psi(z)=\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{z}+O\left(\frac{1}{z^2}\right).\end{gathered} \tag{26.12}$$

where X and Y have the same meaning as in (26.3) (for simplicity, the stresses are assumed to be zero at infinity).

We determine the functions $\overline{\Phi}(z)$ and $\overline{\Psi}(z)$ in the whole plane, with the exception of the cuts, and introduce a new function $\Omega(z)$ defined as

$$\Omega(z)=\overline{\Phi}(z)+z\overline{\Phi}'(z)+\overline{\Psi}(z). \tag{26.13}$$

The function $\Omega(z)$ is analytic in the whole plane, with the exception of the cuts. The combination of stresses $\sigma_y-i\tau_{xy}$ expressed in terms of $\Phi(z)$ and $\Omega(z)$ is of the form

$$\sigma_y-i\tau_{xy}=\Phi(z)+(z-\bar{z})\overline{\Phi}'(\bar{z})+\Omega(\bar{z}). \tag{26.14}$$

On passing to the points of the cuts in (26.14) by a limiting process (from the upper and lower half-planes), we arrive at the Riemann

boundary value problem for $\Phi(z)$ and $\Omega(z)$:

$$\Phi^+(t) + \Omega^-(t) = p(t), \quad \Phi^-(t) + \Omega^+(t) = q(t). \quad (26.15)$$

The functions $p(t)$ and $q(t)$ are assumed to be given. In deriving system (26.15), it has been taken into account, as before, that $\lim_{q\to 0} y\,\overline{\Phi'(z)} = 0$.

System (26.15) reduces to two Riemann boundary value problems for the functions $\Phi(z) + \Omega(z)$ and $\Phi(z) - \Omega(z)$:

$$[\Phi + \Omega]^+ + [\Phi + \Omega]^- = p(t) + q(t), \quad (26.16)$$

$$[\Phi - \Omega]^+ - [\Phi - \Omega]^- = p(t) - q(t). \quad (26.17)$$

The solution of problems (26.16) and (26.17) is obtained in an elementary manner.

Note that practically all results obtained for the case of a half-plane or plane with straight cuts are extended to the case of the inside or outside of a circle and also to the case of a plane with cuts situated along arcs of one circle (see I. N. Kartsivadze [1]).

We shall mention the work of I. A. Prusov [1] dealing with the problem for a plane made up of two bonded planes with different constants when there are arbitrary holes situated strictly in the interior of each region. The problem has been reduced to an auxiliary one for the entire plane by the method of linear relationship.

27. Method of Linear Relationship (Continued)

Consider some special cases of the plane problem where the method of linear relationship is used to advantage.

Let an infinite elastic plane be weakened by a doubly periodic system of cuts parallel to the real axis (see V. Z. Parton [2] and B. A. Kudryavtsev, V. Z. Parton [2]). In the class of doubly periodic problems of the theory of elasticity, most studies are concerned with the equilibrium of plates and shells with circular or elliptical holes (perforated plates and shells). However, similar problems for straight or curvilinear cuts are also of interest in applications (for example, in fracture mechanics) (see V. Z. Parton [1]). Suppose that the primitive period parallelogram has the shape of a rhombus, and the primitive periods ω_1 and ω_2 are conjugate complex numbers. Inside the parallelogram of periods there are two cuts, L_1 and L_2, of the same length situated along a diagonal symmetrically with respect to the centre of the rhombus (Fig. 13). Let a_1, b_1, a_2, b_2 be the co-ordinates of the ends of the cuts:

$$\begin{aligned} a_2 &= \omega_1 + \omega_2 - b_1, & b_2 &= \omega_1 + \omega_2 - a_1, \\ \omega_1 &= a - ib, & \omega_2 &= a + ib. \end{aligned} \quad (27.1)$$

Suppose that a normal load $p(t)$ is given at the edges of the cuts, and the shearing stresses are zero; in consequence, the state of stress in the plane is symmetrical about the x, y axes.

According to the general approach to the solution of problems for a plane with cuts based on the method of linear relationship

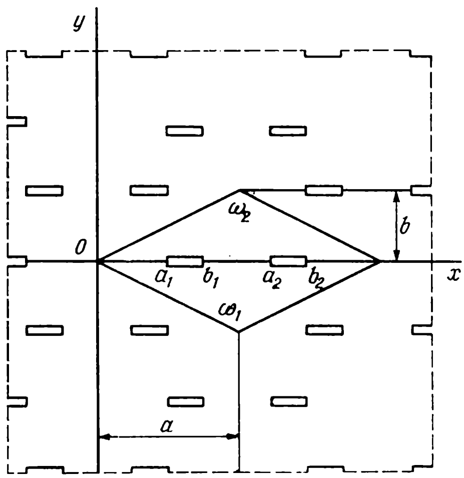

Fig. 13. Half-plane with a doubly periodic system of cuts

(Sec. 26), it is necessary to determine two analytic functions, $\Phi(z)$ and $\Omega(z)$, satisfying some additional conditions which ensure the doubly periodic nature of the state of stress. From the conditions for the double periodicity of the stress components it follows that $\Phi(z)$ is a doubly periodic function and $\Omega(z)$ must satisfy the conditions

$$\begin{aligned}\Omega(z+\omega_1)-\Omega(z)&=(\omega_1-\overline{\omega_1})\,\overline{\Phi'(z)},\\ \Omega(z+\omega_2)-\Omega(z)&=(\omega_2-\overline{\omega_2})\,\overline{\Phi'(z)}.\end{aligned}\tag{27.2}$$

Moreover, from the conditions of mirror symmetry about the x, y axes it follows that

$$\begin{aligned}\Phi(z)&=\overline{\Phi(\bar{z})}, & \Omega(z)&=\overline{\Omega(\bar{z})},\\ \Phi(z)&=\Phi(-z), & \Omega(z)&=\Omega(-z).\end{aligned}\tag{27.3}$$

According to (26.16) and (26.17), we obtain

$$[\Phi(t)+\Omega(t)]^{+}+[\Phi(t)+\Omega(t)]^{-}=2p(t) \text{ on } L, \tag{27.4}$$

$$[\Phi(t)-\Omega(t)]^{+}-[\Phi(t)-\Omega(t)]^{-}=0 \quad \text{on } L. \tag{27.5}$$

Here L is a line of discontinuity in the primitive period parallelogram consisting of the segments L_1 and L_2 of the real axis.

In solving the boundary value problems (27.4) and (27.5), we use the representation of doubly periodic functions as Cauchy-type integrals with a doubly periodic kernel (see L. N. Chibrikova [1]).

Consider an even doubly periodic function

$$F(z)=\frac{1}{2\pi i}\int_L \frac{\rho'(t)\,f(t)\,dt}{\rho(t)-\rho(z)}. \tag{27.6}$$

Taking into account the well-known relations for Weierstrass' functions (see A. M. Zhuravskii [1]), it can be shown that the limiting values of integral (27.6) as the point z approaches the contour L from the left and right are related by equalities similar to formulas (2.9):

$$\begin{gathered} F^+(\tau)-F^-(\tau)=f(\tau), \\ F^+(\tau)+F^-(\tau)=\frac{1}{\pi i}\int_L \frac{\rho'(t)\,f(t)\,dt}{\rho(t)-\rho(\tau)}. \end{gathered} \tag{27.7}$$

In exactly the same way it can be established that for an odd doubly periodic function

$$F_1(z)=\frac{1}{2\pi i}\rho'(z)\int_L \frac{f_1(t)\,dt}{\rho(t)-\rho(z)} \tag{27.6'}$$

the following relations, very useful in applications, are true:

$$\begin{gathered} F_1^+(\tau)-F_1^-(\tau)=f_1(\tau), \\ F_1^+(\tau)+F_1^-(\tau)=\frac{1}{\pi i}\rho'(\tau)\int_L \frac{f_1(t)\,dt}{\rho(t)-\rho(\tau)}. \end{gathered} \tag{27.7'}$$

The functions $\Phi(z)$ and $\Omega(z)$ are sought in the form

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{f(t)\rho'(t)\,dt}{\rho(t)-\rho(z)}, \tag{27.8}$$

$$\Omega(z)=\Phi(z)+\frac{1}{2\pi i}\int_L f(t)\,k(z,\ t)\,dt. \tag{27.9}$$

Here

$$\begin{aligned} k(z,\ t)=(z-t)\,\rho(z-t)-\zeta(z-t)-(z+t)\,\rho(z+t)+\zeta(z+t)-& \\ -\rho_1(z-t)+\rho_1(z+t)-2\zeta(t),& \end{aligned}$$

where the function $\rho_1(z)$ is defined by formula (20.3). The functions $\rho_1(z)$ and $\rho(z)$ satisfy the relations (see E. I. Grigolyuk and L. A. Fil'shtinskii [1])

$$\rho_1(z+\omega_1)-\rho_1(z)=\overline{\omega}_1\rho(z)+\gamma_1,\quad \rho_1(z+\omega_2)-\rho_1(z)=\overline{\omega}_2\rho(z)+\gamma_2.$$

By using these relations, it can easily be shown that the following equalities hold for the function $k(z, t) = k(-z, t)$:

$$k(z+\omega_1, t) = k(z, t) + (\omega_1 - \bar{\omega}_1)\, \rho'(z) \frac{\rho'(t)}{[\rho(t)-\rho(z)]^2},$$

$$k(z+\omega_2, t) = k(z, t) + (\omega_2 - \bar{\omega}_2)\, \rho'(z) \frac{\rho'(t)}{[\rho(t)-\rho(z)]^2}.$$

Thus, conditions (27.4) and (27.5), which must be satisfied by the function $\Omega(z)$, are fulfilled. It is easily seen that the function $k(z, t)$ has no singularities at the points $z = t$, and hence the integral in (27.9) is continuously extendible across the contour L.

Substituting (27.8) and (27.9) in (27.4) and (27.5), it can be verified that condition (27.5) is satisfied identically, and equality (27.4) takes the form

$$\Phi^+(t) + \Phi^-(t) = p(t) - \frac{1}{2\pi i}\int_L f(\tau)\, k(t, \tau)\, d\tau \quad \text{on} \quad L. \tag{27.10}$$

Assuming the right-hand side of (27.10) to be known, we seek the solution of the boundary value problem (27.10) in the class of even doubly periodic functions.

Following the work of L. I. Chibrikova [1], we consider the canonical function of the homogeneous boundary value problem (27.10):

$$X_0(z) = \frac{\sigma(z-A_1)\,\sigma(z-A_2)}{\sqrt{\sigma(z-a_1)\,\sigma(z-b_1)\,\sigma(z-a_2)\,\sigma(z-b_2)}}.$$

Here $\sigma(u)$ is Weierstrass' sigma function, A_1 and A_2 are real constants lying off the line L and satisfying the relation $A_1 + A_2 = \omega_1 + \omega_2$. It is easily seen that $X_0(z)$ is an even elliptic function having two simple zeros in the parallelogram of periods at the points A_1 and A_2. For the limiting values of $X_0(z)$ on the contour L we have the equality $X_0^+(t) + X_0^-(t) = 0$.

Let us transform the expression for $X_0(z)$ using (27.1) and the following formulas (see A. M. Zhuravskii [1]):

$$\rho(u) - \rho(v) = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}, \quad \sigma(u+\omega_1) = -e^{-\delta_1\left(u+\frac{\omega_1}{2}\right)}\sigma(u),$$

$$\sigma(u+\omega_2) = -e^{-\delta_2\left(u+\frac{\omega_2}{2}\right)}\sigma(u), \quad \sigma_1 = 2\zeta\left(\frac{\omega_1}{2}\right), \quad \delta_2 = 2\zeta\left(\frac{\omega_2}{2}\right).$$

The result is

$$X_0(z) = \frac{\sigma^2(A_1)\, e^{1/2\,(2A_1-a_1-b_1)(\delta_1+\delta_2)}}{\sigma(a_1)\,\sigma(b_1)} \quad \frac{[\rho(A_1)-\rho(z)]}{\sqrt{[\rho(a_1)-\rho(z)]\,[\rho(b_1-\rho(z)]}}.$$

If we set $A_1 = 0$ and reject the constant factor, the canonical function of the homogeneous boundary value problem (27.10) may be taken in the form

$$X(z) = \{[\rho(a_1) - \rho(z)]\,[\rho(b_1) - \rho(z)]\}^{-1/2}.$$

The values of $X(z)$ at the upper edge of the cut are determined as follows

$$X^+(x) = i\{[\rho(a_1)-\rho(x)][\rho(x)-\rho(b_1)]\}^{-1/2} \quad \text{with} \quad a_1<x<b_1,$$
$$X^+(x) = -i\{[\rho(a_1)-\rho(x)][\rho(x)-\rho(b_1)]\}^{-1/2} \quad \text{with} \quad a_2<x<b_2.$$

With the help of the canocical function $X(z)$ we can find the general solution of the homogeneous problem (27.10) for an even doubly periodic function $\Phi_0(z)$, which satisfies the condition

$$\Phi_0(z) = \overline{\Phi_0(\bar{z})} \tag{27.3'}$$

and has no poles inside the parallelogram of periods with the cut L. This solution is

$$\Phi_0(z) = X(z)[\beta+\beta_1\rho(z)]. \tag{27.11}$$

Here β and β_1 are arbitrary constants. By virtue of condition (27.3′) and the equality $\rho(z) = \overline{\rho(\bar{z})}$ the constants β and β_1 are real quantities.

By using the canonical function $X(z)$, we obtain (according to Sec. 6) a boundary condition following from (27.10):

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{p(t)}{X^+(t)} - \frac{1}{2\pi i}\frac{1}{X^+(t)}\int_L f(\tau)k(t,\tau)\,d\tau. \tag{27.12}$$

On the basis of formulas (27.6) and (27.7) we obtain the solution of the boundary value problem (27.12):

$$\Phi(z) = \Phi_*(z) - $$
$$-\frac{1}{2\pi i}X(z)\int_L \frac{\rho'(t)}{X^+(t)[\rho(t)-\rho(z)]}\left[\frac{1}{2\pi i}\int_L f(\tau)k(t,\tau)\,d\tau\right]dt. \tag{27.13}$$

Here

$$\Phi_*(z) = \frac{1}{2\pi i}X(z)\int_L \frac{p(t)\rho'(t)\,dt}{X^+(t)[\rho(t)-\rho(z)]} + \frac{1}{2}\Phi_0(z); \tag{27.14}$$

in the case of a constant load applied at the edges of the cut

$$\Phi_*(z) = \frac{p}{2} - \frac{p}{4}[\rho(a_1)+\rho(b_1)-2\rho(z)]X(z) + \frac{1}{2}\Phi_0(z). \tag{27.15}$$

According to expressions (27.6) to (27.8), the boundary values of $\Phi(z)$ taken from the left and right of L satisfy the relation $\Phi^+(t) - \Phi^-(t) = f(t)$; substituting the limiting values (27.13) in this relation, we find

$$f(u) = [\Phi_*^+(u) - \Phi_*^-(u)] +$$
$$+\frac{1}{2\pi^2}X^+(u)\int_L \frac{\rho'(t)}{X^+(t)[\rho(t)-\rho(u)]}\left[\int_L f(\tau)k(t,\tau)\,d\tau\right]dt. \tag{27.16}$$

Because of the continuity of $k(t, \tau)$ the order of integration in the last term on the right-hand side can be interchanged. By making this interchange, we obtain a Fredholm integral equation for determining the function $f(t)$:

$$f(u) = [\Phi_*^+(u) - \Phi_*^-(u)] + \int_L f(\tau) K(u, \tau)\, d\tau. \qquad (27.17)$$

Here

$$K(u, \tau) = \frac{1}{2\pi^2} X^+(u) \int \frac{k(t, \tau)\, \rho'(t)\, dt}{X^+(t)[\rho(t) - \rho(u)]}.$$

By using relations (27.8) and (27.9), we determine the stress components on the real axis

$$(\sigma_y - i\tau_{xy})_{y=0} = 2\Phi(x) + \frac{2}{2\pi i} \int_L f(t)\, k(x, t)\, dt.$$

To find the constants β and β_1, we consider the resultant vector of all forces acting along an arc AB joining two congruent points. The expression for the resultant vector is of the form $[\varphi'(z) = \Phi(z),$ $\omega'(z) = \Omega(z)]$

$$X + iY = -ig(z)\Big|_A^B = -i[\varphi(z) + (z - \bar{z})\overline{\Phi(z)} + \omega(\bar{z})]_A^B.$$

If the external load on each of the cuts is self-balanced, the resultant vector of all forces along the arc AB is zero, i.e.,

$$g(z + \omega_1) - g(z) =$$
$$= \varphi(z + \omega_1) - \varphi(z) + \omega(\bar{z} + \bar{\omega}_1) - \omega(\bar{z}) + (\omega_1 - \bar{\omega}_1)\overline{\Phi(z)} = 0,$$
$$g(z + \omega_2) - g(z) =$$
$$= \varphi(z + \omega_2) - \varphi(z) + \omega(\bar{z} + \bar{\omega}_2) - \omega(\bar{z}) + (\omega_2 - \bar{\omega}_2)\overline{\Phi(z)} = 0.$$

Since the function $\Phi(z)$ is even and doubly periodic, the following relations hold for the function $\varphi(z)$: $\varphi(z + \omega_1) - \varphi(z) = c_1$, $\varphi(z + \omega_2) - \varphi(z) = c_2$. Here $c_1 = 2\varphi(\omega_1/2)$, $c_2 = 2\varphi(\omega_2/2)$. Similarly, the following conditions are obtained from equalities (27.2) for the function $\omega(z)$:

$$\omega(z + \omega_1) - \omega(z) = (\omega_1 - \bar{\omega}_1)\Phi(z) + d_1,$$
$$\omega(z + \omega_2) - \omega(z) = (\omega_2 - \bar{\omega}_2)\Phi(z) + d_2,$$
$$d_1 = 2\omega\left(\frac{\omega_1}{2}\right) - (\omega_1 - \omega_2)\Phi\left(\frac{\omega_1}{2}\right),$$
$$d_2 = 2\omega\left(\frac{\omega_2}{2}\right) + (\omega_1 - \omega_2)\Phi\left(\frac{\omega_2}{2}\right).$$

By using the above relations, from the condition that the resultant vector of all forces along the arc AB is zero we obtain $c_1 + d_1 = 0$,

$c_2 + d_2 = 0$. Substituting the values of the constants c_1, c_2, d_1, d_2, we find

$$\varphi\left(\frac{\omega_1}{2}\right) + \omega\left(\frac{\omega_1}{2}\right) - \frac{1}{2}(\omega_1 - \omega_2)\,\Phi\left(\frac{\omega_1}{2}\right) = 0,$$

$$\varphi\left(\frac{\omega_2}{2}\right) + \omega\left(\frac{\omega_2}{2}\right) + \frac{1}{2}(\omega_1 - \omega_2)\,\Phi\left(\frac{\omega_2}{2}\right) = 0.$$

Consequently, the constants β and β_1 must be chosen so as to fulfil these equalities.

We shall now describe a somewhat different procedure for solving the boundary value problems (27.4) and (27.5). The functions $\Phi(z)$ and $\Omega(z)$ are sought in the form

$$\Phi(z) = \Phi_0(z) + \sum_{j=1}^{\infty} \Phi_j(z), \tag{27.18}$$

$$\Omega(z) = \Phi_0(z) + \sum_{j=1}^{\infty} \left\{\Phi_j(z) + \frac{1}{2\pi i}\int_L [\Phi_{j-1}^+(t) - \Phi_{j-1}^-(t)]\, k(z, t)\, dt\right\}. \tag{27.19}$$

Here Φ_j ($j = 0, 1, 2, \ldots$) are even doubly periodic functions. Obviously, conditions (27.4) and (27.5) are fulfilled if the functions Φ_j ($j = 0, 1, 2, \ldots$) satisfy the following relations on the contour L:

$$\Phi_0^+(t) + \Phi_0^-(t) = p(t) \quad (t \in L), \tag{27.20}$$

$$\Phi_j^+(t) + \Phi_j^-(t) = -\frac{1}{2\pi i}\int_L [\Phi_{j-1}^+(\tau) - \Phi_{j-1}^-(\tau)]\, k(t, \tau)\, d\tau \quad (j = 1, 2, \ldots). \tag{27.21}$$

Thus, we have a sequence of boundary value problems for the doubly periodic even functions $\Phi_j(z)$. By determining the functions $\Phi_j(z)$ ($j = 0, 1, 2, \ldots$) from (27.20) and (27.21), we obtain, in the case of constant loads along the length of a cut,

$$\Phi_0(z) = \frac{p}{2\pi i} X(z) \int_L \frac{\rho'(t)\, dt}{X^+(t)\,[\rho(t) - \rho(z)]} + \frac{1}{2} X(z)\,[\beta^{(0)} - \beta_1^{(0)}\rho(z)], \tag{27.22}$$

$$\Phi_j(z) = -\frac{1}{2\pi i} X(z) \int_L \frac{\rho'(t)}{X^+(t)\,[\rho(t) - \rho(z)]} \times \left\{\frac{1}{2\pi i}\int_L [\Phi_{j-1}^+(\tau) - \Phi_{j-1}^-(\tau)]\, k(t, \tau)\, d\tau\right\} dt + \frac{1}{2} X(z)\,[\beta^{(j)} + \beta_1^{(j)}\rho(z)] \quad (j = 1, 2, \ldots). \tag{27.23}$$

The conditions that the resultant vector of all forces is zero may be written as

$$\varphi_0\left(\frac{\omega_1}{2}\right)+\omega_0^*\left(\frac{\omega_1}{2}\right)=0,\quad \varphi_0\left(\frac{\omega_2}{2}\right)+\omega_0^*\left(\frac{\omega_2}{2}\right)=0,\tag{27.24}$$

$$\begin{gathered}\varphi_j\left(\frac{\omega_1}{2}\right)+\omega_j^*\left(\frac{\omega_1}{2}\right)=\frac{1}{2}(\omega_1-\omega_2)\,\Phi_{j-1}\left(\frac{\omega_1}{2}\right),\\ \varphi_j\left(\frac{\omega_2}{2}\right)+\omega_j^*\left(\frac{\omega_2}{2}\right)=-\frac{1}{2}(\omega_1-\omega_2)\,\Phi_{j-1}\left(\frac{\omega_2}{2}\right)\\ (j=1,\,2,\,\ldots).\end{gathered}\tag{27.25}$$

Here

$$\varphi_j(z)=\int_0^z \Phi_j(z_1)\,dz_1\qquad (j=0,\,1,\,2,\,\ldots),$$

$$\omega_0^*(z)=\varphi_0(z),\quad \omega_j^*(z)=\int_0^z \Omega_j(z_1)\,dz_1,\qquad (j=1,\,2,\,\ldots),\tag{27.26}$$

$$\Omega_j(z)=\Phi_j(z)+\frac{1}{2\pi i}\int_L\left[\Phi_{j-1}^+(t)-\Phi_{j-1}^-(t)\right]k(z,\,t)\,dt.$$

Relations (27.24) to (27.26) enable one to determine successively the constants $\beta^{(0)}$, $\beta_1^{(0)}$, $\beta^{(j)}$, $\beta_1^{(j)}$.

To obtain numerical results, use has been made of the approximate solution of the problem taking into account the first terms in expansions (27.18) and (27.19). The normal stresses on the real axis for $x<a_1$ are determined as follows:

$$\begin{aligned}\sigma_y\Big|_{\substack{y=0\\ x<a_1}}&\approx 2\Phi_0(x)=\\ &=p-\frac{p}{2}\,\frac{2\rho(x)-\rho(a_1)-\rho(b_1)}{\sqrt{[\rho(x)-\rho(a_1)]\,[\rho(x)-\rho(b_1)]}}+\\ &\qquad+\frac{\beta^{(0)}+\beta_1^{(0)}\rho(x)}{\sqrt{[\rho(x)-\rho(a_1)]\,[\rho(x)-\rho(b_1)]}}.\end{aligned}\tag{27.27}$$

From conditions (27.24) we find

$$\begin{gathered}p\left[\frac{\omega_1}{2}-I_1\left(\frac{\omega_1}{2}\right)\right]+\frac{1}{2}\{p\,[\rho(a_1)+\rho(b_1)]+\\ +2\beta^{(0)}\}\,I_0\left(\frac{\omega_1}{2}\right)+\beta_1^{(0)}I_1\left(\frac{\omega_1}{2}\right)=0,\\ p\left[\frac{\omega_2}{2}-I_1\left(\frac{\omega_2}{2}\right)\right]+\frac{1}{2}\{p\,[\rho(a_1)+\rho(b_1)]+\\ +2\beta^{(0)}\}\,I_0\left(\frac{\omega_2}{2}\right)+\beta_1^{(0)}I_1\left(\frac{\omega_2}{2}\right)=0.\end{gathered}\tag{27.28}$$

Here

$$I_0\left(\frac{\omega_1}{2}\right)=I_0'+iI_0''=\int_0^{1/2\omega_1} X(z)\,dz,$$

$$I_1\left(\frac{\omega_1}{2}\right)=I_1'+iI_1''=\int_0^{1/2\omega_1} \rho(z)\,X(z)\,dz,$$

$$I_0\left(\frac{\omega_2}{2}\right)=I_0'-iI_0'',\quad I_1\left(\frac{\omega_2}{2}\right)=I_1'-iI_1''.$$

From Eqs. (27.28), the values of the constants $\beta^{(0)}$ and $\beta_1^{(0)}$ are

$$\begin{aligned}\beta^{(0)}&=-\frac{1}{2}\,p\,[\rho(a_1)+\rho(b_1)]+\frac{1}{2}\,p\,\frac{aI_1''+bI_1'}{I_0''I_1'-I_0'I_1''}\\ \beta_1^{(0)}&=p-\frac{1}{2}\,p\,\frac{bI_0'+aI_0''}{I_0''I_1'-I_0'I_1''}.\end{aligned}\tag{27.29}$$

Here $a=\frac{1}{2}(\omega_1+\omega_2)$, $b=\frac{1}{2i}(\omega_2-\omega_1)$. In accordance with (27.27) we find the approximate value of the stress intensity factor

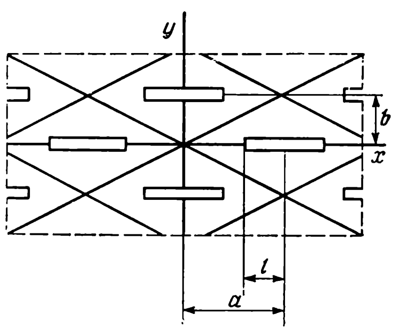

Fig. 14. Half-plane with a doubly periodic system of cuts for $b_1 = a_2$

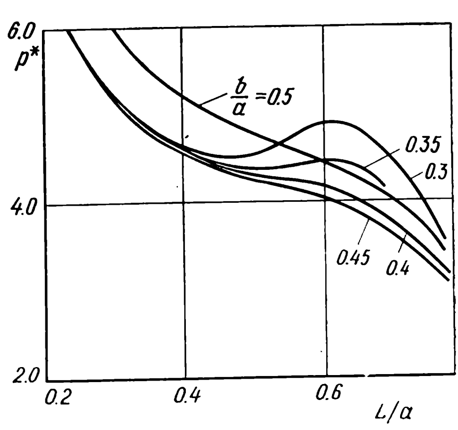

Fig. 15. Relation between the critical stress p^* and the crack length (l/a) for several values of b/a

at the point a_1:

$$K_I = \lim_{x \to a_1} [\sqrt{2\pi (a_1 - x)}\, \sigma_y (x, 0) \mid_{x < a_1}] \approx$$

$$\approx -\sqrt{\frac{\pi}{2}} \frac{p [\rho (a_1) - \rho (b_1)] - 2\beta^{(0)} - 2\beta_1^{(0)} \rho (a_1)}{\sqrt{-\rho'(a_1) [\rho (a_1) - \rho (b)_1]}}. \quad (27.30)$$

The application of Irwin's fracture condition ($K_I = K_{Ic}$, K_{Ic} is the critical stress intensity factor) makes it possible to relate the crack length and the applied loads. The calculations by formula (27.30) have been performed for the case $b_1 = a_2$ (there is one cut inside the parallelogram of periods; see Fig. 14). Figure 15 shows the relation between the value $p^* = p \sqrt{2\pi a}/K_{Ic}$ and the relative length of the cut l/a for several values of b/a. It follows from the above solution that the development of a system of cracks may be stable for some values of the ratio b/a (mutual strengthening of cracks) (see V. Z. Parton [1], V. Z. Parton, E. M. Morozov [1]).

Chapter VI

INTEGRAL EQUATIONS FOR FUNDAMENTAL THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY

28. Generalized Elastic Potentials

We present the theory of generalized elastic potentials following, for the most part, the classical theory of harmonic potentials.

Consider a space filled with an elastic medium having Lamé's constants λ and μ. Let a force $\varphi\,[\varphi_1(q), \varphi_2(q), \varphi_3(q)]$ be applied at a point $q\,(y_1, y_2, y_3)$. As noted in Sec. 14, the displacements at an arbitrary point $p\,(x_1, x_2, x_3)$ are then expressed as the product of the Kelvin-Somigliana matrix $\Gamma(p, q)$ and the vector $\varphi(q)$:

$$U(p) = \Gamma(p, q)\,\varphi(q). \tag{28.1}$$

We fix some plane at the point p by assigning the direction of its normal $n\,(n_1, n_2, n_3)$. The stress vector on this plane corresponding to the displacement field (28.1) is then represented as the result of applying the operator $T_{n(p)}$ (14.7) to displacements (28.1).

As a result, as shown in Sec. 14, we arrive at the expression for the stress vector in the form of the product of the matrix $\Gamma_1(p, q)$ (14.22) and the vector $\varphi(q)$. Below is the expression for the elements of the matrix $\Gamma_1(p, q)$ written out in full:

$$\Gamma_{1(kj)}(p, q) = -\left[m\delta_{kj} + 3n\frac{(x_k - y_k)(x_j - y_j)}{r^2}\right]\frac{\sum\limits_{l=1}^{3}(x_l - y_l)\,n_l(p)}{r^3} + \\ + m\left[n_k(p)\frac{(x_j - y_j)}{r^3} - n_j(p)\frac{(x_k - y_k)}{r^3}\right]. \tag{28.2}$$

Suppose, now, that forces $\varphi(q)$ are given on some closed Lyapunov surface* S; the function $\varphi(q)$ is assumed to belong to the class

* A surface is called a *Lyapunov surface* if the following conditions are fulfilled: (1) at each point of the surface S there exists a definite normal (tangent plane); (2) there is a number $d > 0$ such that no straight line parallel to the normal at any point q of the surface S cuts in more than one point a part S_q of the surface S lying inside a sphere of radius d centred at q; (3) the angle

H-L. The total displacements in the entire space are then representable in the form of an integral:

$$U(p) = \int_S \Gamma(p, q)\, \varphi(q)\, dS_q; \tag{28.3}$$

this integral is called the *generalized elastic potential for a simple layer*. It is obvious that potential (28.3) satisfies Lamé's equations in both the region D^+ and the region D^-. It is worth speaking of the *limiting values of the simple layer potential from both the inside of the surface* (from the region D^+) *and the outside* (from the region D^-) and also of the so-called *direct value* obtained by direct substitution of points of the surface S in the integrand. However, by using the estimates employed in similar investigations in the theory of a simple layer harmonic potential, it can be shown that all three values coincide (see, for example, L. N. Sretenskii [1]). Consequently, the *generalized elastic potential for a simple layer is a continuous vector function in the entire space.*

Let now to each point p situated in a sufficiently thin layer enclosing the surface S be related in a one-to-one manner a point q' of the surface S in such a way that the normal to the surface at the point q' passes through that point. This circumstance enables values of the stress vector to be properly determined at interior points of the regions D^+ and D^- situated in the layer mentioned above. The stress vector is represented by an integral, namely

$$T_{n(q')} V(p) = \int_S \Gamma_1(p, q)\, \varphi(q)\, dS_q. \tag{28.4}$$

By $n(q')$ is meant the direction of the normal to the surface S at a point q' corresponding to a point p in the sense indicated above.

We now use the matrix $\Gamma_2^{\mathrm{I}}(p, q)$ (14.28) to construct a generalized elastic potential called the *potential for a double layer of the first kind.** This potential is given by

$$W_2^{\mathrm{I}}(p) = \int_S \Gamma_2^{\mathrm{I}}(p, q)\, \varphi(q)\, dS_q. \tag{28.5}$$

This potential, just as the simple layer potential, is a function satisfying Lamé's equations outside and inside the surface.

Note that the matrix $\Gamma_2^{\mathrm{I}}(p, q)$ may be assigned a physical mean-

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$\gamma(q, q') = (\overline{n_q}, \overline{n_{q'}})$ formed by the normals at points q and q' satisfies the following equation: $\gamma(q, q') < Ar^{\delta}$, where r is the distance between the points q and q', A and δ are some constants, and $0 < \delta \leqslant 1$.

* In Sec. 14 this was taken into account in introducing the corresponding indexing.

ing. The product of this matrix and a vector $\varphi(q)$ represents a displacement in the entire space produced by a concentrated moment $\varphi(q)$ applied at a point q in a plane with normal $\boldsymbol{n}$.

The direct value of the double layer potential $W_2^1(p)$ can only be understood as a principal value (see Sec. 7) since the elements ω_{ij} (28.2) have a second-order pole.

To study the limiting values of displacements represented by the generalized elastic potential for a double layer of the first kind, we must first consider the simplest case when the density is constant. Denote it by φ_0 and refer to formula (14.27) assuming that the vector φ_0 is the displacement of the whole body and the point p is situated in the region D^+. The first term on the right-hand side of the equality vanishes, and in consequence we obtain the following expression:

$$2\varphi_0 = -\int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q)\,\varphi_0\,dS_q. \tag{28.6}$$

If the point p is taken in the region D^-, we arrive at the equality

$$\int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q)\,\varphi_0\,dS_q = 0. \tag{28.7}$$

We now turn to the calculation of the direct (singular) value of the double layer potential when the point p is situated on the surface S. Let this point be surrounded by a sphere σ_ε of small radius ε, and let parts of its surface situated in the regions D^+ and D^- be denoted by σ_ε^+ and σ_ε^-, respectively. Denote by S_ε^* a part of the surface S situated outside the sphere σ_ε.

It follows from the preceding discussion that the integral over the surface $S_\varepsilon^* + \sigma_\varepsilon^-$ is equal to $-2\varphi_0$, and over the surface $S_\varepsilon^* + \sigma_\varepsilon^+$ it is zero. From considerations of symmetry we conclude that in the limit as the radius ε decreases to zero, the integrals over the surfaces σ_ε^+ and σ_ε^- are equal (apart from the sign). Consequently, the integral over the surface S_ε^* is equal to $-\varphi_0$ in the limit:

$$\int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q)\,\varphi_0\,dS_q = -\varphi_0 \quad (p \in S). \tag{28.8}$$

It will be recalled that it was in this way that the singular value of an integral was defined in Sec. 7. The above result represents Gauss' theorem (in the generalized form).

Sometimes in the literature formulas (28.6) to (28.8) are written symbolically in an alternate form (E is the unit matrix):

$$\int_S \Gamma_2^{\mathrm{I}}(p, q)\, dS_q = -2E \quad (p \in D^+),$$

$$\int_S \Gamma_2^{\mathrm{I}}(p, q)\, dS_q = 0 \quad (p \in D^-),$$

$$\int_S \Gamma_2^{\mathrm{I}}(p, q)\, dS_q = -E \quad (p \in S).$$

In this case the integrals are understood as the integrals of each element of the matrix. The above results enable us to establish the limiting theorems for the double layer potential of the first kind. We transform the expression for the potential as

$$W_2^{\mathrm{I}}(p, q) = \int_S \Gamma_2^{\mathrm{I}}(p, q)\,[\varphi(q) - \varphi(q')]\, dS_q + \\ + \varphi(q') \int_S \Gamma_2^{\mathrm{I}}(p, q)\, dS_q, \qquad (28.9)$$

where q' is a fixed point on the surface S. As the point p approaches the point q' (from the inside or outside of the surface S), the first term in (28.9) is a continuous function; the behaviour of the second term has been studied above. The foregoing result may now be stated as the following theorem.

If the limiting values of the double layer potential from the inside and outside are denoted by $W_2^{\mathrm{I}+}(q')$ and $W_2^{\mathrm{I}-}(q')$, respectively, and the direct (singular) value by $W_2^{\mathrm{I}}(q)$, the resulting conclusion may be written as follows:

$$W_2^{\mathrm{I}+}(q') - W_2^{\mathrm{I}-}(q') = -2\varphi(q'), \quad W_2^{\mathrm{I}+}(q') + W_2^{\mathrm{I}-}(q') = 2W_2^{\mathrm{I}}(q'). \qquad (28.10)$$

Relations (28.10) may be rewritten in an alternate (equivalent) form:

$$W_2^{\mathrm{I}+}(q') = -\varphi(q') + W_2^{\mathrm{I}}(q'), \qquad W_2^{\mathrm{I}-}(q') = \varphi(q') + W_2^{\mathrm{I}}(q'). \qquad (28.10')$$

Note that the results obtained above are equally valid for potentials generated by the matrix $\Gamma_2^{\mathrm{II}}(p, q)$ (14.33), which are called *double layer potentials of the second kind.* Equalities (28.6) to (28.8), (28.10), (28.10′) hold since the matrix $\Gamma_2^{\mathrm{II}}(p, q)$ differs from the matrix $\Gamma_2^{\mathrm{I}}(p, q)$ by terms whose integral vanishes. By analogy, this potential is denoted by $W_2^{\mathrm{II}}(p, q)$.

Consider another potential introduced by H. Weyl [1] and called the *antenna potential** in which the kernel is taken to be the solution of the third kind $M(p, q)$ (14.36):

$$A(p, q) = \frac{1}{2\pi} \int_S M(p, q)\, \varphi(q)\, dS_q. \tag{28.11}$$

Potential (28.11) is a continuous function inside the surface S if the density function $\varphi(q)$ is continuous. As to the physical meaning of the antenna potential, it corresponds to the solution of an elasticity problem obtained by superimposing the solutions for a half-space loaded by a concentrated force on its surface (Boussinesq's solution).

We now turn to the study of the behaviour of the stress operator T_n of the simple layer potential (28.3). It is obvious that direct substitution of the points of the surface S in (28.4) leads to an integral that must be understood as a singular one. We introduce into consideration the limiting values of the stress operator from the inside and outside and denote them by T_n^+V and T_n^-V, respectively.

We transform expression (28.4) assuming that the point p approaches a point q' of the surface:

$$\lim_{p\to q'} T_{n(q')} V(p) = \lim_{p\to q} \left\{ \int_S [\Gamma_1(p, q) + \Gamma_2^{\mathrm{I}}(p, q)]\, \varphi(q)\, dS_q - \int_S \Gamma_2^{\mathrm{I}}(p, q)\, \varphi(q)\, dS_q \right\}. \tag{28.12}$$

By introducing a local co-ordinate system, and making estimates essentially similar to those for a harmonic potential (see S. L. Sobolev [2]), it can be shown that the first term varies in a continuous manner when the point q crosses the surface moving along a normal to it. The behaviour of the last term has been previously studied. Consequently, there exist *direct* $T_nV(q')$ and *limiting values of the stress operator*, and they are related by

$$T_n^+V(q') - T_n^-V(q') = 2\varphi(q'), \quad T_n^+V(q') + T_n^-V(q') = 2T_nV(q'), \tag{28.13}$$

$$T_n^+V(q') = \varphi(q') + T_nV(q'), \quad T_n^-V(q') = -\varphi(q') + T_nV(q'). \tag{28.13'}$$

Operations similar to those given above show that the following equality holds for the limiting values (from the inside) of the stress

* The name antenna potential comes from the fact that the function $\frac{\partial v}{\partial x} = \ln(r + x) = \int \frac{dx}{r}$, where v is defined by (14.34), is the electrostatic potential due to charges uniformly distributed along a normal to the surface S of an antenna.

operator of the antenna potential:

$$T_n^+ A(q') = -\varphi(q') + \frac{1}{2\pi} \int_S T_n M(q', q)\, \varphi(q)\, dS_q. \qquad (28.14)$$

The expression for the matrix itself is

$$T_{n(q')} M(q', q) =$$

$$= -3 \begin{Vmatrix} \left(\frac{\partial r}{\partial x_1}\right)^2 & \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} \\ \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & \left(\frac{\partial r}{\partial x_2}\right)^2 & \frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_3} \\ \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_3} & \frac{\partial r}{\partial x_2}\frac{\partial r}{\partial x_3} & \left(\frac{\partial r}{\partial x_3}\right)^2 \end{Vmatrix} \frac{d}{dn(q')}\frac{1}{r} + 3\frac{m}{n} E. \qquad (28.15)$$

Consider the behaviour of the limiting values of the stress operator of the double layer potential, $T_n^+ W_2^{\mathrm{I}}$ and $T_n^- W_2^{\mathrm{I}}$. As in the theory of a harmonic potential, here only some sufficient conditions for the existence of these limiting values have been obtained. There is, for example, the following analogue of Lyapunov's theorem.*

Let the density $\varphi(q)$ of the double layer potential of the first kind be such that there exists a limiting value of the operator $T_n W_2^{\mathrm{I}}$ from one side of the surface; then *there exists a limiting value of the operator $T_n W_2^{\mathrm{I}}$ from the other side*, and *these limiting values coincide*. To prove the theorem, suppose that there exists a limiting value of the operator $T_n^+ W_2^{\mathrm{I}}$, which is denoted by $F(q)$ for convenience. This means that there is a function $U_1(p)$ in the region D^+ satisfying Lamé's equations, for which there exists a limiting value of the operator T_n^+ equal to $F(q)$. We denote the value of this function on the surface by $U_1(q)$ and form the double layer potential:

$$W_2^{\mathrm{I}}(p) = W_2^{\mathrm{I}}(p, U_1) = \int_S \Gamma_2^{\mathrm{I}}(p, q)\, U_1(q)\, dS_q.$$

We now introduce a new function $w(p)$ defined in the region D^+ by the formula $w(p) = U_1(p) - \frac{1}{2} W_2^{\mathrm{I}}(p)$, and in the region D^- by the formula $w = -\frac{1}{2} W_2^{\mathrm{I}}(p)$.

This new function is continuous in the entire space by virtue of property (28.10). Note that the procedure of constructing the function $U_1(p)$ is that of solving the second fundamental problem, and this is accomplished by means of the simple layer potential (see Sec. 31). As has been previously proved, there exist limiting values of the stress operator for the simple layer potential. Hence, the dis-

* Known as the *Lyapunov-Tauber theorem*.

placement $U_1(p)$ also has this property, the limiting value coinciding with the function $F(q)$.

We further refer to Betti's formula (14.27) and write it as

$$U_1(p)=\frac{1}{2}\int_S \Gamma_2^{\mathrm{I}}(p,q)\,U_1(q)\,dS_q-\frac{1}{2}\int_S \Gamma(p,q)\,F(q)\,dS_q=$$

$$=\frac{1}{2}W_2^{\mathrm{I}}(p)-\frac{1}{2}\int_S \Gamma(p,q)\,F(q)\,dS_q \quad (p\in D^+), \qquad (28.16)$$

Consequently, the function $w(p)$ (28.6) is represented in the region D^+ as a simple layer potential:

$$w(p)=-\frac{1}{2}\int_S \Gamma(p,q)\,F(q)\,dS_q. \qquad (28.17)$$

It follows from (28.17) that the sum $w(p)+\frac{1}{2}\int_S \Gamma(p,q)\,F(q)\,dS_q$ vanishes in the region D^+. But, as proved above, the function $w(p)$ is continuous. The second term is also continuous since it is a simple layer potential. Consequently, this sum is identically zero in the region D^-, too (because of the uniqueness of the solution of the problem I^-). We thus arrive at the following representation of the function $W_2^{\mathrm{I}}(p)$ in the entire space:

$$W_2^{\mathrm{I}}(p)=2U_1(p)+\int_S \Gamma(p,q)\,F(q)\,dS_q \quad (p\in D^+), \qquad (28.18)$$

$$W_2^{\mathrm{I}}(p)=\int_S \Gamma(p,q)\,F(q)\,dS_q \quad (p\in D^-). \qquad (28.18')$$

Each of the right-hand sides of equalities (28.18) and (28.18′) has limiting values of the stress operator. Hence, the potential itself also has this property. A direct calculation of the operator T_n from both sides of the surface leads to the required equality. In the case when the existence of a limiting value of the operator T_n^- is assumed, the proof is carried out in a similar way.

Note that the properties of the double layer potential thus obtained provide an elementary way of solving a practically important problem of negative allowance. Suppose that an elastic medium is situated both outside and inside a Lyapunov surface S, and the following conditions are fulfilled on the boundary:

$$U^+(q)-U^-(q)=F_1(q),\quad T_n^+U(q)=T_n^-U(q).$$

From the above properties of the double layer potential of the first kind it follows that the solution of this problem (assuming that the limiting values of the operator T_n exist) is representable as a double

layer potential:

$$W_2^{\mathrm{I}}(p)=\frac{1}{2}\int_S \Gamma_2^{\mathrm{I}}(p,\ q)\,F_1(q)\,dS_q. \tag{28.19}$$

This approach can obviously be extended to include the case when the total region occupied by the two elastic bodies is different from a complete space. In this case the difference between the required displacements and potential (28.19) leads to a new problem for a continuous body with properly modified boundary conditions on the outer surfaces. It should be noted that a similar plane problem solved in Sec. 22 has required considerably greater efforts.

29. Regular and Singular Integral Equations for Fundamental Three-dimensional Problems

The foregoing generalized elastic potentials for a simple layer, a double layer of the first kind, a double layer of the second kind and the antenna potential enable one to construct the corresponding integral equation. In general, to solve either of the problems (the first or second) we can, in principle, use any one of the potentials introduced above since the displacement equations of elastic equilibrium (Lamé's equations) are identically satisfied. It is desirable, however, that the resulting equations should have a favourable structure, they should belong to the classes of integral equations of the second kind. It follows from this condition that the solution of the first fundamental problem (problem I) must be sought in the form of a double layer potential of the first or second kind (otherwise integral equations of the first kind are obtained).

Let $\boldsymbol{f}(q)$ be a given boundary value of displacements on the surface S. If the solution of the problem is sought in the form of the generalized double layer potential of the first kind (28.5), we obtain, by formulas (28.10′), integral equations for the interior and the exterior problem, respectively. We represent these equations at once in the standard form for integral equations of the second kind by introducing an auxiliary parameter ν:

$$\varphi(q)-\nu\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi(q')\,dS_{q'}=\boldsymbol{F}(q). \tag{29.1}$$

The value $\nu=1$ corresponds to the interior problem (I^+), and the value $\nu=-1$ to the exterior problem (I^-). The function $\boldsymbol{F}(q)$ coincides with the function $\boldsymbol{f}(q)$ for the exterior problem, and with the function $-\boldsymbol{f}(q)$ for the interior problem. Equation (29.1) is singular since the matrix $\Gamma_2^{\mathrm{I}}(q,\ q')$ has terms with a second-order singularity (ω_{ij}).

If the solution is sought in the form of a double layer potential of the second kind, the resulting integral equations are entirely sim-

ilar in appearance (the so-called *Lauricella equations*):

$$\varphi(q) - \nu \int_S \Gamma_2^{II}(q, q') \varphi(q') \, dS_{q'} = F(q). \tag{29.2}$$

The essential difference between Eqs. (29.1) and (29.2) is that, as mentioned above, the first one is singular, and the second is regular.

We now turn to the consideration of the second fundamental problem (problem II). Let stresses $f(q)$* be given on the surface S. If this problem is solved by means of a double layer potential of the first or second kind, we obtain some functional equations whose solvability is not studied at all. In a sense these quations cannot even be called integral since the interchanging of the order of integration and the stress operator is ruled out.

We seek the solution of the second problem in the form of a simple layer potential. The integral equation that follows from (28.13′) can conveniently be written as

$$\varphi(q) - \nu \int_S \Gamma_1(q, q') \varphi(q') \, dS_{q'} = F(q). \tag{29.3}$$

Here the value $\nu = 1$ corresponds to the exterior problem (II^-), and the value $\nu = -1$ to the interior problem (II^+). The function $F(q)$ is equal to $f(q)$ in the interior problem, and to $-f(q)$ in the exterior problem. The integral equation (29.3) is a singular integral equation of the second kind.

Similar equations (though regular) are obtained when the solution is constructed on the basis of the antenna potential. These equations, which follow directly from the limiting equality (28.14), are of the form

$$\varphi(q) + \nu \int_S T_{n(q')} M(q', q) \varphi(q') \, dS_q = F(q), \tag{29.4}$$

where $\nu = -1$ (problem II^+) and the surface S is convex.**

We now turn to the question of the solvability of the equations obtained***. Suppose that the Fredholm alternative applies to the signular equations introduced above (for proof see Sec. 30).

Consider the integral equations (29.1) and (29.3) for $\nu = 1$. These singular equations are companion to each other, and hence the question of their solvability is decided simultaneously. Suppose that these homogeneous equations have non-trivial solutions and let $\varphi_0(q)$

* For generality of writing we retain the notation used in the first problem.

** Of course, the results of Ya. B. Lopatinskii [1], which enable one to reduce boundary value problems for elliptic systems to regular integral equations, are readily extended to the equations of the theory of elasticity.

*** Equations (29.1) and (29.3) have been proposed by V. D. Kupradze [2].

be the corresponding solution of the equation for the problem II^-. We resort to the simple layer potential $V(p, \varphi_0)$. This function solves the elasticity problem with zero stresses on the surface S, the displacements at infinity being of order $1/R$, and the stresses of order $1/R^2$. By the uniqueness theorem, such a solution for an infinite region is identically zero. On the other hand, the potential $V(p, \varphi_0)$ is a continuous function in the entire space. Hence, the displacement field generated by it in the region D^+ vanishes on the surface S. For the first fundamental problem, it follows from the uniqueness theorem that these displacements [i.e., the potential $V(p, \varphi_0)$] are zero in the entire region D^+. The non-trivial solution introduced above must be equal to half the difference of the limiting values of the operator $T_nV(p, \varphi_0)$ at the points of the surface S [according to (28.13)], and therefore it is zero.

Thus, the integral equations (29.1) and (29.3) do not have the number $\nu = 1$ for an eigenvalue. On the basis of the Fredholm alternatives we conclude that the singular integral equations for the problems I^+ and II^- are solvable for arbitrary right-hand sides.

The integral equations have been derived above using one or another representation for the required displacement vector. We now give another method of constructing integral equations. Let us use Betti's formula for an arbitrary vector function $U(p)$ satisfying Lamé's equations in the region D^+. Suppose that stresses $T_nU(q)$ are prescribed on the surface S. The second integral in (14.27) may then be considered known; denote it by $\Phi(p)$. By applying a limiting process in (14.27), we pass to the points of the surface S from the inside. According to (28.10′), we obtain, after collecting like terms, an integral equation for the displacements $U(p)$ on the surface S:

$$U(q) + \int_S \Gamma_2^{\mathrm{I}}(q, q')\, U(q')\, dS_{q'} = \Phi(q). \tag{29.5}$$

If the analogous formula (14.30) for the displacement in the region D^- is used, then by the same reasoning we arrive at the integral equation

$$U(q) - \int_S \Gamma_2^{\mathrm{I}}(q, q')\, U(q')\, dS_{q'} = \Phi(q). \tag{29.5′}$$

It is obvious that the two equations may be written in a unified form by introducing a parameter ν:

$$U(q) - \nu \int_S \Gamma_2^{\mathrm{I}}(q, q')\, U(q')\, dS_{q'} = \Phi(q). \tag{29.6}$$

The value $\nu = 1$ corresponds to the problem II^-, and the value $\nu = -1$ to the problem II^+. As seen, this equation is completely identical with Eq. (29.1).

A direct derivation of integral equations for the first fundamental problem from Betti's formulas leads to an integral equation of the first kind with a regular kernel. A singular equation of the second kind can be obtained by applying the stress operator to all terms of Betti's formula, and assuming that its limiting values for a double layer potential with a given density function $U(q)$ exist. The resulting equation coincides with Eq. (29.3).

In the works of T. A. Cruse [1-3] preference is given to singular integral equations obtained from Betti's formulas. The author assumes that these equations are also applicable to piecewise smooth surfaces since Betti's identities hold for such equations. When passing to the points of the surface by a limiting process, however, it is necessary to use the formulas for a double layer potential, and they are only valid for Lyapunov surfaces. It is quite a different matter that the actual solution of the problem may be found preferable because of the greater smoothness of the boundary condition.

We now turn to the analysis of the equations when $\nu = -1$, this corresponding to the problems I^+ and II^-. It has been shown in Sec. 14 that there exists a non-trivial solution of the interior problem of the theory of elasticity with zero stress values; it corresponds to a rigid displacement of the surface of the body and is represented in Cartesian co-ordinates as

$$U_1 = a_1 + qx_3 - rx_2,\ U_2 = a_2 + rx_1 - px_3,$$
$$U_3 = a_3 + px_2 - qx_1, \tag{29.7}$$

where a_1, a_2, a_3, p, q, r are arbitrary constants, and there can be no other solutions. Let this displacement be denoted by $U_0(p)$. It is obvious that the corresponding stresses are zero in the entire region D^+. We shall try to represent the displacement $U_0(p)$ in the form of a double layer potential of the first kind extended over the surface S. By using formula (14.27) in reference to the displacement $U_0(p)$, we obtain

$$U_0(p) = -\frac{1}{2}\int_S \Gamma_2^{\mathrm{I}}(p, q)\, U_0(q)\, dS_q. \tag{29.8}$$

The first term in (14.27) vanishes since $T_n U_0(p) \equiv 0$. By applying a limiting process on the left-hand and right-hand sides of relation (29.8), we pass to the points of the surface S from the inside. According to formula (28.10′), the right-hand side becomes

$$\frac{1}{2} U_0(q') - \frac{1}{2}\int_S \Gamma_2^{\mathrm{I}}(q', q)\, U_0(q)\, dS_q.$$

Thus, it follows from (29.8) that the vector $U_0(q)$ must satisfy the equation

$$U_0(q') + \int_S \Gamma_2^{\mathrm{I}}(q', q) U_0(q) dS_q = 0, \tag{29.9}$$

which is the integral equation for the problem I^-.

It is obvious that the displacements $U_0(p)$ characterized by six independent constants $(a_1, a_2, \ldots, a_6)$ may be represented in some way or other as a set of six linearly independent solutions. We may proceed, for example, as follows. Assume in succession only one of the six constants to be different from zero; then corresponding to each such variant there is a distinct linearly independent solution of Eq. (29.9) [denoted by ψ_k^* $(k = 1, 2, \ldots, 6)$].

It follows from Fredholm's theorems that the companion equation (the equation for the problem II^+) has at least six linearly independent solutions. We denote them by $\psi_k(q)$ $(k = 3, 2, \ldots, 6)$ and prove that they form a complete system of linearly independent solutions of the equations for the problem II_0^+.* Let $\psi_0(q)$ be one more solution linearly independent of the six solutions introduced above. Consider the simple layer potentials

$$V(p, \psi_0) = \int_S \Gamma(p, q) \psi_0(q) dS_q,$$

$$V_k(p, \psi_k) = \int_S \Gamma(p, q) \psi_k(q) dS_q.$$

These potentials solve the problem II_0^+ with zero stress values on the boundary, and hence they must represent a rigid displacement. Consequently, the potential $V(p, \psi_0)$ is a linear combination of the $V_k(p, \psi_k)$:

$$V(p, \psi_0) = \sum_{k=1}^{6} C_k V_k(p, \psi_k). \tag{29.10}$$

We rewrite this equality in an alternate form:

$$\int_S \Gamma(p, q) \left[\psi_0(q) - \sum_{k=1}^{6} C_k \psi_k(q)\right] dS_q = 0 \quad (p \in D^+).$$

Let the bracketed expression be denoted by $\psi(q)$. The potential $V(p, \psi)$ is then identically zero in the region D^+. By the argument used previously it can be shown that this potential is zero in the region D^-, which proves the linear dependence of the function $\psi_0(q)$ on the functions $\psi_k(q)$. From this it follows that the system $\psi_k(q)$

* The additional subscript zero indicates that the right-hand side is zero.

is complete, and on the basis of (29.8) we conclude that all double layer potentials $W(p, \varphi_k) = 0$ $(p \in D^-)$.

We now turn to the proof of the existence of the solution of the integral equation for the problem II$^+$. According to Fredholm's third theorem, the necessary and sufficient condition for the solvability of the non-homogeneous equation

$$\boldsymbol{\varphi}(q) - \nu \int\limits_S \boldsymbol{\Gamma}_1(q, q')\boldsymbol{\varphi}(q')\, dS_{q'} = \boldsymbol{F}(q) = \boldsymbol{f}(q) \tag{29.3}$$

when $\nu = -1$ is that the right-hand side $\boldsymbol{f}(q)$ should be orthogonal to the complete system of eigenfunctions $\boldsymbol{\varphi}_k^*(q)$ of the companion equation:

$$\int\limits_S \boldsymbol{f}(q)\boldsymbol{\varphi}_k^*(q)\, dS_q = 0 \quad (k = 1, 2, \ldots, 6). \tag{29.11}$$

These conditions have a definite mechanical meaning, which becomes clear if we use the form of writing the functions $\boldsymbol{\varphi}_k^*(q)$ introduced above [see (29.7)] assuming in succession any one of the constants to be different from zero. In this case conditions (29.11) take the form

$$\begin{gathered} \int\limits_S f_i(q)\, dS_q = 0, \\ \int\limits_S \{f_i x_{i+1} - f_{i+1} x_i\}\, dS_q = 0, \\ \int\limits_S \{f_i x_{i+2} - f_{i+2} x_i\}\, dS_q = 0, \end{gathered} \tag{29.12}$$

where f_i $(i = 1, 2, 3)$ are the components of the vector $\boldsymbol{f}(q)$ and $x_4 = x_1$, $x_5 = x_2$, $f_4 = f_1$, $f_5 = f_2$. It is obvious that formulas (29.12) express the conditions that the resultant vector and the resultant moment of the forces applied to the surface are zero. Note that the solution of the integral equation (29.3) when $\nu = -1$ is not unique. The complete solution of this equation is represented as the sum

$$\boldsymbol{\varphi}(q) = \sum_{k=1}^{6} C_k \boldsymbol{\psi}_k(q) + \boldsymbol{\varphi}^*(q),$$

where C_k are arbitrary constants and $\boldsymbol{\varphi}^*(q)$ is a particular solution of the non-homogeneous equation. The simple layer potentials $V(p, \boldsymbol{\psi}_k)$ are a solution of the problem II$^+$ for zero values of the stress vector. By the uniqueness theorem, these potentials represent a rigid-body displacement and have no effect on the stresses.

Thus, (29.11) are not only conditions for the solvability of Eq. (29.3) (determined by the very representability of the solution in a

specially chosen form in terms of a simple layer potential), but also conditions for the solvability of the original physical problem.

In contrast to the problem II^+, the conditions for the solvability of the equation for the problem I^- have no physical meaning and only determine the representability of the solution in terms of a double layer potential. It can be shown that the double layer potential decreases at infinity as R^{-2}. Meantime the solutions of problems decrease at infinity as R^{-1} in general.

If the conditions for the orthogonality of the right-hand side to the eigenfunctions of the companion equation are not fulfilled, then, in accordance with the general procedure,* additional terms are introduced into the representation of displacement. It is simplest to take them in the form of concentrated forces applied in the region D^+, their magnitudes being determined from the orthogonality conditions. It can be shown that the resulting system of equations is always solvable. The practical implementation of this approach presents difficulties since it requires a knowledge of the eigenfunctions $\boldsymbol{\psi}_k(q)$ $(k = 1, 2, \ldots, 6)$.

We now turn to the investigation of the integral equations obtained on the basis of Betti's formulas. Since these equations are completely identical** with Eqs. (29.1) and (29.3) (with appropriate rearrangement), the foregoing analysis is fully extended to these equations.

The problem II^+ requires an additional analysis since the right-hand side is complicated and the orthogonality conditions are not explicit. The integral equation is identical with Eq. (29.1) for the problem I^- and may be written as

$$\boldsymbol{U}(q) + \int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(q, q')\,\boldsymbol{U}(q')\,dS_{q'} = \int_S \boldsymbol{\Gamma}(q, q')\,\boldsymbol{f}(q')\,dS_{q'}. \qquad (29.13)$$

The necessary and sufficient conditions for the solvability of this problem are

$$\int_S \boldsymbol{\psi}_k(q) \int_S \boldsymbol{\Gamma}(q, q')\,\boldsymbol{f}(q')\,dS_{q'}\,dS_q = 0 \quad (k = 1, 2, \ldots, 6), \qquad (29.14)$$

where $\boldsymbol{\psi}_k(q)$ are the eigenfunctions of the companion equation. It has been shown above, however, that the potentials $V(p, \boldsymbol{\psi}_k)$ are the rigid displacement vectors. By interchanging the order of integra-

* By analogy with the Dirichlet problem.
** The right-hand sides of the equations will come under consideration later.

tion in (29.14), we therefore obtain

$$\int_S \psi_k(q) \int_S \boldsymbol{\Gamma}(q, q') \boldsymbol{f}(q')\, dS_{q'} dS_q = \\ = \int_S \boldsymbol{V}(q', \psi_k) \boldsymbol{f}(q')\, dS_{q'} = 0. \qquad (29.15)$$

Consequently, the conditions for the solvability of the integral equation (29.13) are, as before, the conditions for the existence of the solution of the original physical problem, and hence, by the formulation, they are automatically fulfilled.

We now turn to the analysis of regular integral equations. Let us solve the first fundamental problem by using the generalized elastic potential for a double layer of the second kind. The corresponding equation is rewritten as

$$\boldsymbol{\varphi}(q) - \nu \int_S \boldsymbol{\Gamma}_2^{\mathrm{II}}(q, q') \boldsymbol{\varphi}(q')\, dS_{q'} = \boldsymbol{F}(q) \qquad (29.16)$$

The value $\nu = 1$ corresponds to the problem I^+, and the value $\nu = -1$ to the problem I^-. The function $\boldsymbol{F}(q)$ coincides with given displacements $\boldsymbol{f}(q)$ in the exterior problem, and is equal to their negative in the interior problem.

Consider the problem I^+ and prove that the corresponding equation is always solvable. Otherwise the companion homogeneous equation

$$\boldsymbol{\psi}(q) - \int_S \boldsymbol{\Gamma}_2^{\mathrm{II}}(q', q) \boldsymbol{\psi}(q')\, dS_{q'} = 0 \qquad (29.17)$$

would have a non-trivial solution. Suppose that $\boldsymbol{\psi}_0(q)$ is such a solution of Eq. (29.17), and form a simple layer potential with the density function $\boldsymbol{\psi}_0(q)$. Condition (29.17) means that the limiting value of N-operator (14.19) [for $\alpha = \mu(\lambda + \mu)/(\lambda + 3\mu)$] of this potential from the inside is zero. But the potential itself solves the elasticity problem for the region D^- with a given zero value of the operator N on the surface.* By the uniqueness theorem proved in Sec. 14, this potential is zero in the region D^-. But a simple layer potential is a continuous function. It is therefore zero in the region D^+, too. Consequently, the density of this potential, which is proportional to the jump in the limiting values of the operator N, is identically zero.

By a reasoning similar to that used above, we find that since the solution of the interior problem is not unique, the value $\nu = -1$ is an eigenvalue of Eq. (29.16).

The study of the integral equations (29.16) with $\nu = -1$ can be carried out successfully only when the eigenfunctions of the companion equations are known (the analogy with the exterior Dirichlet

* It will be recalled that this problem has no physical meaning, but its formulation is necessary for the proof of the solvability of Lauricella's equations.

problem is quite appropriate here). The general formal procedure of solution is completely similar to the foregoing solution of the problem I^- by means of a double layer potential of the first kind.

The question of the solvability of the integral equation (29.4) is left open. Of course, from the fact that the problem is solvable only if the body is in equilibrium it automatically follows that the number $\nu = -1$ is an eigenvalue. But it is quite possible that these conditions are inadequate.

Naturally, the results obtained here and in Sec. 28 are, in fact fully extended to the plane problem of the theory of elasticity. At the same time it should be noted that the integral equations given in Secs. 18 and 19 do not admit a direct generalization to the three-dimensional case, and that is why they have been considered in a separate chapter.

The starting point for constructing the corresponding theory is the solution of the problem of displacements at a point p_1 in the plane under a force φ (φ_1, φ_2) applied at a point p (y_1, y_2). Consider a second-order matrix

$$\widetilde{\mathbf{\Gamma}}(p_1, \quad p) = \frac{\lambda+\mu}{2\pi\mu(\lambda+2\mu)} \left\| \begin{matrix} \frac{\lambda+3\mu}{\lambda+\mu}\ln r - \left(\frac{\partial r}{\partial x_1}\right)^2 & -\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} \\ -\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & \frac{\lambda+3\mu}{\lambda+\mu}\ln r - \left(\frac{\partial r}{\partial x_2}\right)^2 \end{matrix} \right\| \quad (29.18)$$

corresponding to the Kelvin-Somigliana matrix (14.21) and often called the *Boussinesq matrix*. N. S. Kakhniashvili [1] uses the fundamental solution (29.18) to construct a similar theory of the plane problem. The author has formed simple and double layer potentials of the first and second kind. The corresponding limiting theorems have been obtained for the case of smooth contours and a density function belonging to the class H-L.

The integral equations for the second fundamental problem similar to Eq. (29.3) may be written as

$$(\varphi q') - \lambda \int_L \widetilde{\mathbf{\Gamma}}_1(q', \ q)\,\varphi(q)\,ds_q = \boldsymbol{F}(q') \qquad (29.19)$$

where the matrix $\widetilde{\mathbf{\Gamma}}_1(q', \ q) = \boldsymbol{T}_{n(q')}\widetilde{\mathbf{\Gamma}}(q', \ q)$ is

$$\widetilde{\mathbf{\Gamma}}(q', \ q) = \left\| \begin{matrix} m_1 + n_1\left(\frac{\partial r}{\partial x_1}\right)^2 & n_1\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} \\ n_1\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} & m_1 + n_1\left(\frac{\partial r}{\partial x_2}\right)^2 \end{matrix} \right\| \frac{d}{dn\,(q')}\ln r + \\ + m_1 \left\| \begin{matrix} 0 & \omega_{12}(q', \ q) \\ -\omega_{12}(q', \ q) & 0 \end{matrix} \right\| \ln r.$$

Here

$$m_1 = \frac{\mu}{\pi(\lambda+2\mu)}, \qquad n_1 = \frac{2(\lambda+\mu)}{\pi(\lambda+2\mu)},$$

$$\omega_{12} = \frac{\partial}{\partial x_2}\cos(n_{q'}, x_1) - \frac{\partial}{\partial x_1}\cos(n_{q'}, x_2).$$

As in the three-dimensional case, the elements of matrix (29.18) are doubled.

We shall mention some new features. Here there is a complete clarity in respect to the Lyapunov-Tauber theory. If the density function together with its derivative satisfies the H-L condition, there exist the limiting values of the stress operator and they are equal (from the two sides). In the case of the exterior problem II^-, for the solution to be regular at infinity, the resultant vector of the forces must vanish. The question of the index of the resulting system of one-dimensional singular equations is solved on the basis of the well-known results of the theory of systems of this kind (see N. P. Vekua [1]). A direct calculation shows that this index is zero, and hence the Fredholm alternatives apply to the systems of singular integral equations of the plane problem in elasticity, and this in combination with the uniqueness theorems enables one to answer the question of their solvability.

As in the three-dimensional case, the integral equations for the second fundamental problem can be constructed on the basis of Betti's formula (in its two-dimensional form) (see, for example, F. J. Rizzo [1]):

$$\boldsymbol{U}(q') + \int_L \widetilde{\boldsymbol{\Gamma}}_2(q', q)\boldsymbol{U}(q)\,dS = \int_L \widetilde{\boldsymbol{\Gamma}}(q', q)\boldsymbol{F}(q)\,dS_q. \quad (29.20)$$

Here $\widetilde{\boldsymbol{\Gamma}}_2(q', q)$ denotes a matrix companion to $\widetilde{\boldsymbol{\Gamma}}_1(q', q)$.

In conclusion it might be well to point out the research specially devoted to the construction and investigation of (regular and singular) integral equations for axially symmetric problems and also to the development of methods for their solution. We shall mention the investigations of A. Ya. Aleksandrov [2-4], Yu. D. Kopeikin [3], T. Kermandis [1], G. N. Polozhii [1, 2], Yu. I. Solov'ev [1-4], D. I. Sherman [1] and others.

30. Extension of the Fredholm Alternatives to Singular Integral Equations of the Theory of Elasticity

In the preceding section we have obtained singular integral equations for the first and second fundamental problems of the theory of elasticity. Each of Eqs. (29.3) and (29.6) represents a system of singular equations. In these equations, the singular part of the kernel

(of the ijth element) is of the form, apart from a factor [see (14.22)]:

$$\frac{1}{2\pi}\left[\frac{(x_i-y_i)\,n_j-(x_j-y_j)\,n_i}{r^3}\right]. \tag{30.1}$$

It will be recalled that if the variables are changed, the argument of each element of the symbolic determinant undergoes a linear transformation; the argument of the symbolic determinant undergoes the same transformation, and in consequence the set of its values is invariant under a change of variables (see Sec. 7). We therefore introduce local co-ordinates at each point q of the surface bounding the elastic body with the x_1 and x_2 axes taken in the tangent plane and the x_3 axis along the normal. The unknowns are chosen to be the components of vectors $\boldsymbol{\varphi}(q)$ in the local co-ordinate system (φ_1, φ_2, φ_3), which also leaves the index of the system unaltered. The system of equations (29.6), for example, is then written as

$$\begin{gathered}
\varphi_1(q) \pm A\int_S \frac{(x_1-y_1)}{r^3}\,\varphi_3(q')\,dS_{q'}+\boldsymbol{T}_1(q)=\boldsymbol{F}_1(q),\\
\varphi_2(q) \pm A\int_S \frac{(x_2-y_2)}{r^3}\,\varphi_3(q')\,dS_{q'}+\boldsymbol{T}_2(q)=\boldsymbol{F}_2(q),\\
\varphi_3(q) \pm A\int_S \frac{(x_1-y_1)\,\varphi_1(q')+(x_2-y_2)\,\varphi_2(q')}{r^3}\,dS_{q'}+\boldsymbol{T}_3(q)=\boldsymbol{F}_3(q).
\end{gathered} \tag{30.2}$$

Here $\boldsymbol{T}_i(q)$ are regular integral operators with a singularity $r^{\alpha-2}$ acting on the functions $\varphi_1(q)$, $\varphi_2(q)$, $\varphi_3(q)$ (α is the Lyapunov index of the surface S), A is a constant.

The characteristics of the singular integrals appearing in Eqs. (30.2) are $(x_1-y_1)/r=\cos\theta$ and $(x_2-y_2)/r=\sin\theta$. The symbols for these characteristics are obtained by multiplying them by $2\pi i$ (with the argument θ replaced by λ).

We now write out the symbolic determinant:

$$\begin{vmatrix} 1 & 0 & iA\cos\theta \\ 0 & 1 & iA\sin\theta \\ -iA\cos\theta & -iA\sin\theta & 1 \end{vmatrix} = 1-A^2=\frac{3-4\sigma}{4(1-\sigma)^2}. \tag{30.3}$$

This determinant is different from zero for values of Poisson's ratio of interest in the theory of elasticity.

Since the symbolic matrix is Hermitian ($\sigma_{ij}=\bar{\sigma}_{ji}$), Fredholm's theorems are fulfilled for Eq. (29.6). A similar conclusion can be drawn with regard to Eq. (29.3).

It is natural that a direct regularization of integral equations of elasticity is unnecessary, if only because of the difficulties involved in constructing the corresponding equivalent system of Fredholm equations. Note also that V. G. Maz'ya and V. D. Sapozhnikova [1] give a constructive expression for the regularizing operator.

31. Spectral Properties of Regular and Singular Integral Equations. Method of Successive Approximations

The theorems, proved in Sec. 29, on the solvability of regular and singular integral equations for the fundamental three-dimensional problems of elasticity completely solve, in principle, the problem of the mathematical justification of the integral equation method. However, in actually solving these equations (for example, Lauricella's equations for problems I^+) by the mechanical quadrature method, the problem is reduced to a system of algebraic equations, frequently of very high order. At the same time we remark that the solution of two-dimensional singular equations by the mechanical quadrature method requires a corresponding mathematical justification similar to that given in Sec. 12 for the case of one-dimensional equations.

A very promising method for the solution of the equations for the fundamental three-dimensional problems is the method of successive approximations. Its computational advantages will be discussed in detail in Sec. 33. Here we only note that its justification is equally valid for both regular and singular equations and follows from the spectral properties of these equations (see Sec. 1).

We now prove that in a circle of unit radius centred at the zero, the point $\nu = -1$ is the only eigenvalue, and it is a first-order pole of the resolvent. We begin the corresponding analysis with the consideration of Lauricella's integral equation (see Pham The Lai [1])

$$\varphi(q)-\nu\int_S \Gamma_2^{II}(q,\ q')\,\varphi(q')\,dS_{q'}=f(q) \tag{31.1}$$

whose companion equation is of the form

$$\psi(q)-\nu\int_S \Gamma_2^{II}(q',\ q)\,\psi(q')\,dS_{q'}=g(q). \tag{31.2}$$

This equation may be interpreted as an integral equation obtained by using a simple layer potential $V(p,\psi)$, namely

$$V(p,\psi)=\int_S \Gamma(p,\ q)\,\psi(q)\,dS_q$$

for the solution of the boundary value problem with a given value of the N-operator on the surface S.

Similarly to (28.13), we write the equalities

$$NV^+-NV^-=2\psi(q),\quad NV^++NV^-=2\int_S \Gamma_2^{II}(q,\ q')\,\psi(q')\,dS_{q'}. \tag{31.3}$$

Equation (31.2) then becomes

$$(1-\nu)\,[NV^{+}]-(1+\nu)\,[NV^{-}]=2g\,(q). \qquad (31.4)$$

By applying generalized Betti's formula (14.14) to the displacement $V\,(p,\,\psi)$ in the region D^{+}, we obtain

$$\int_S V\,(q,\,\psi)\,NV^{+}dS_q=-\int_{D^+} E\,(V,\,V)\,d\Omega, \qquad (31.5)$$

where $E\,(V,\,V)$ is defined by formula (14.17).

In a similar way, the formula for the displacements in the region D^{-} is obtained as

$$\int_S V\,(q,\,\psi)\,NV^{-}dS_q=-\int_{D^-} E\,(V,\,V)\,d\Omega. \qquad (31.6)$$

The last two formulas show that the left-hand side in (31.5) is always negative, and in (31.6) it is always positive.

We assign two continuous functions, $\psi_a\,(q)$ and $\psi_b\,(q)$, on the surface S and, regarding them as density functions, we form simple layer potentials $V\,(p,\,\psi_a)$ and $V\,(p,\,\psi_b)$:

$$V\,(p,\,\psi_a)=\int_S \mathbf{\Gamma}\,(p,\,q)\,\psi_a\,(q)\,dS_q, \qquad (31.7)$$

$$V\,(p,\,\psi_b)=\int_S \mathbf{\Gamma}\,(p,\,q)\,\psi_b\,(q)\,dS_q. \qquad (31.8)$$

On the basis of the generalized formula (14.15) we have

$$\int_S \{V_aN^{+}V_b-V_bN^{+}V_a\}\,dS_q=0,\ \int_S \{V_aN^{-}V_b-V_bN^{-}V_a\}\,dS_q=0. \quad (31.9)$$

We now prove that all poles of the resolvent of Eq. (31.1) are real. Suppose that there exists a complex root $\nu_0=a+ib$ and consider the corresponding solution of the homogeneous equation (31.2) $\nu_0\,(q)=\nu_a^0+i\nu_b^0$. By using the functions ν_a^0 and ν_b^0, we form simple layer potentials $V\,(p,\,\nu_a^0)$ and $V\,(p,\,\nu_b^0)$, respectively, and make a change in Eq. (31.4) (the appropriate change consists in replacing ν by ν_0):

$$(1-\nu_0)\,[N^{+}V_a+iN^{+}V_b]=(1+\nu_0)\,[N^{-}V_a+iN^{-}V_b]. \quad (31.10)$$

We multiply both sides of the last equality by V_a-iV_b and integrate over the surface S using formulas (31.9). The result is

$$(1-\nu_0)\int_S \{V_aN^{+}V_a+V_bN^{+}V_b\}dS=$$
$$=(1+\nu_0)\int_S \{V_aN^{-}V_a+V_bN^{-}V_b\}\,dS. \quad (31.11)$$

Since the first integral is different from zero when $\nu_a^0 \neq 0$ and $\nu_b^0 \neq 0$, it follows that the ratio $(1 - \nu_0)/(1 + \nu_0)$ is real, and consequently $b = 0$. From equality (31.11) it also follows that the ratio $(1 - \nu_0)/(1 + \nu_0)$ is negative (or zero). Hence, the value of ν_0 must not be numerically less than unity. It will be recalled that the investigation made in Sec. 29 has shown that the value $\nu = 1$ is not a pole of the resolvent while $\nu = -1$ is a pole.

The last result of spectral theory is the proof of the fact that all poles are simple.

Let ν_1 be a pole of order other than unity. Then [see (1.29) and (1.30)] there exist two functions on the surface S, which will be denoted by φ_a and φ_b, satisfying the following equalities:

$$\varphi_a(q) = \nu_1 \int_S \Gamma_2^{II}(q', q)\, \varphi_a(q')\, dS_{q'},$$

$$\varphi_b(q) - \varphi_a(q)/\nu_1 = \nu_1 \int_S \Gamma_2^{II}(q', q)\, \varphi_b(q')\, dS_{q'}.$$

We represent these equalities in terms of potentials V_a and V_b determined by φ_a and φ_b as density functions:

$$\begin{gathered} N^+V_a - N^-V_a = \nu_1 [N^+V_a + N^-V_a], \\ N^+V_a - N^-V_a + N^+V_b - N^-V_b = \nu_1 [N^+V_b + N^-V_b]. \end{gathered} \tag{31.12}$$

By multiplying the first equality by V_b and the second by V_a, adding together, and integrating, we obtain, noting (31.9),

$$\int_S V_a N^+ V_a\, dS = \int_S V_b N^- V_b\, dS. \tag{31.13}$$

According to (31.5) and (31.6), the expressions appearing on the left-hand and right-hand sides of equality (31.13) must be of opposite sign (the expression on the right-hand side is different from zero). Thus, we come to a contradiction, and hence all poles of the resolvent are simple.

It follows from what has been proved above that the integral equation (29.2) can be solved by the method of successive approximations when $\nu = 1$ using the solution in the form of (10.7) and (10.9).

We now turn to the consideration of singular integral equations. We rewrite the singular equations for the fundamental three-dimensional problems of the theory of elasticity: in the case of the first fundamental problem

$$\varphi(q) - \nu \int_S \Gamma_2^{I}(q, q')\, \varphi(q')\, dS_{q'} = F(q), \tag{29.1}$$

in the case of the second fundamental problem

$$\boldsymbol{\varphi}(q)-\nu\int_S \boldsymbol{\Gamma}_1(q,\ q')\,\boldsymbol{\varphi}(q')\,dS_{q'}=\boldsymbol{F}(q), \tag{29.3}$$

$$\boldsymbol{U}(q)-\nu\int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}(q')\,dS_{q'}=\boldsymbol{\Phi}(q). \tag{29.6}$$

Since Eqs. (29.1) and (29.3) are companion, and Eqs. (29.1) and (29.6) are identical (except for the values of the right-hand sides and the unknown functions), it is advisable to consider their spectral properties simultaneously.

By repeating much of the analysis described above in reference to Lauricella's equations with the operator $\boldsymbol{N}$ accordingly replaced by the operator $\boldsymbol{T}$ [and, of course, with generalized Betti's formula (14.14) replaced by the conventional one], it can be shown that all these singular equations have only real eigenvalues numerically not less than unity. The values $\nu = 1$ and $\nu = -1$ have been previously considered in Sec. 29.

It is obvious that the proof of the applicability of the Fredholm alternatives to Eq. (29.3) given in Sec. 28 is extended to the other equations. As shown in Sec. 30 from the existence of equivalent regularization it follows that the solutions of the singular equations can be represented by means of the resolvent. The study of the behaviour of the resolvent in the neighbourhood of the point $\nu = -1$ [carried out in the same way as for the regular equation (29.2), i.e., by means of equalities (1.29) and (1.30)] shows (V. D. Kupradze [3]) that this point is a simple pole of the resolvent, and the coefficient multiplying $1/(\nu + 1)$ in its expansion is the solution of the companion homogeneous equation (as a function of the argument q).

It follows from the above discussion that the singular integral equations for the fundamental problems of the theory of elasticity can be solved by the method of successive approximations, with the exception of the equations for the problem I^- (since the conditions for their solvability cannot generally be established). This fact has been pointed out by Pham The Lai [1].

Equation (29.1) for $\nu = 1$ (problem I^+) can be solved by using series (10.8) or (10.10). Equations (29.3) and (29.6) should be solved in a similar way in the case of the problem II^-. In the problem II^+ the original series (10.2) is convergent.

We now turn to a direct solution of the above equations. Let us first consider a singular equation, obtained by combination* of

* Since they are identical (in the proper notation).

(29.1) and (29.6), and Eq. (29.3):

$$\varphi(q)-\nu\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi(q')\,dS_{q'}=F(q), \qquad (31.15)$$

$$\varphi(q)-\nu\int_S \Gamma_1(q,\ q')\,\varphi(q')\,dS_{q'}=F(q). \qquad (29.3)$$

The implementation of the method of successive approximations in reference to Eqs. (31.15) and (29.3) consists in evaluating integrals of the form

$$\varphi_n(q)=\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi_{n-1}(q')\,dS_{q'},$$

$$\varphi_n(q)=\int_S \Gamma_1(q,\ q')\,\varphi_{n-1}(q')\,dS_{q'}. \qquad (31.16)$$

These integrals are singular, and hence the application of the well-known cubature formulas for their evaluation is ruled out. P. I. Perlin [7, 9, 10] has suggested that the following identities should be used:

$$\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi(q')\,dS_{q'}=-\varphi(q)+\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,[\varphi(q')-\varphi(q)]\,dS_{q'}, \qquad (31.17)$$

$$\int_S \Gamma_1(q,\ q')\,\varphi(q')\,dS_{q'}=-\varphi(q)+\int_S \{\Gamma_1(q,\ q')\,\varphi(q')-\Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi(q)\}\,dS_{q'}; \qquad (31.18)$$

they have been termed regular representations of singular integrals since their right-hand sides are improper integrals* [when the function $\varphi(q)$ belongs to the class H-L]. The derivation of these identities is based on the method of reducing singularities, proposed by L. V. Kantorovich [1], using equality (28.8) and the fact that the singular terms of the matrices $\Gamma_2^{\mathrm{I}}(q,\ q')$ and $\Gamma(q,\ q')$ are the same.

By applying these representations to evaluate the right-hand sides of relations (31.1), we arrive at recurrence relations

$$\varphi_n(q)=-\varphi_{n-1}(q)+\int_S \Gamma_2^{\mathrm{I}}(q,\ q')\,[\varphi_{n-1}(q')-\varphi_{n-1}(q)]\,dS_{q'}. \qquad (31.17')$$

$$\varphi_n(q)=-\varphi_{n-1}(q)+\int_S \{\Gamma_1(q,\ q')\,\varphi_{n-1}(q')-\Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi_{n-1}(q)\}\,dS_{q'}. \qquad (31.18')$$

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* Note that for the integral with the kernel $\Gamma_2^{\mathrm{II}}(q',\ q)$ we have an identity similar to (31.17), which can be used to improve the efficiency of calculations.

In the work of V. M. Likhovtsev and P. I. Perlin [1] the following question is discussed. Let an equation be solved when the boundary conditions are different from zero only over a certain small part of the bounding surface. The use of the foregoing recommendations is not advisable since it may be stated with sufficient certainty that the unknown density functions will differ only slightly from zero values on the surface (enclosing the zone of loading). In solving integral equations, it is therefore advantageous to perform the integration only over the remainder of the surface. It is then necessary to introduce new regular representations since representations (31.17) and (31.18) are based on identity (28.8), which is only valid for a closed surface.

Consider a procedure suggested by N. V. Kurnosov and V. M. Likhovtsev [1]. Denote by S_1 the part of the surface S that will be used in calculations. Formula (28.8) is rewritten as

$$\int_{S_1} \Gamma_2^{\mathrm{I}}(q, q')\, dS_{q'} = -E - \int_{S-S_1} \Gamma_2^{\mathrm{I}}(q, q')\, dS_{q'}.$$

A recurrence relation similar to (31.17′) is of the form*

$$\varphi_n(q) = -\varphi_{n-1}(q)\left[E + \int_{S-S_1} \Gamma_2^{\mathrm{I}}(q, q')\, dS_{q'}\right] + \\ + \int_{S_1} \Gamma_2^{\mathrm{I}}(q, q')\left[(\varphi_{n-1}(q') - \varphi_{n-1}(q)\right] dS_{q'}. \qquad (31.19)$$

The additional integral is regular since it is determined only at the points of the surface S_1. No questions arise in evaluating this integral along the line bounding the surface S_1 since, first, it is advisable to carry out the calculations only at the interior points of S_1 and, second, the corresponding densities at the edge points must be very small (by the formulation of the problem); hence, the error involved in evaluating the integral comes to naught.

The construction of a recurrence relation corresponding to (31.18′) is now obvious.

The representations introduced above are also useful in solving problems for semi-infinite regions. Naturally, by drawing an auxiliary surface of sufficiently large size inside or outside the body, it is possible to pass to the corresponding interior or exterior problem. If the original boundary conditions are required to be self-balanced, it appears unnecessary to perform the integration over the entire closed surface in solving the integral equation, and this involves the use of recurrence relations of the form of (31.19).

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* A special case when the surface S_1 is flat has been considered in the work of V. M. Likhovtsev and P. I. Perlin [1] mentioned above.

32. Differential Properties of Solutions of Integral Equations and Generalized Elastic Potentials

In formulating the boundary value problems of the theory of elasticity it was required that the solution should be regular (i.e., it should have continuous first derivatives in the closed region $\bar{D}$ and continuous second derivatives in the open region D). From the foregoing results it follows that if the bounding surface belongs to the class of Lyapunov surfaces and if the boundary conditions belong to the class H-L, then the solution of integral equations belongs to the class L_2. It is necessary to answer the last question, namely whether the simple or double layer potentials now generated are regular functions (in the sense noted above). This is a very complicated mathematical question. We therefore restrict ourselves to some definitions and the statement of basis results.* A function $\varphi(p)$ is said to belong to the class $C^k(\bar{D})$ if at each point of the region D it has all derivatives up to order k, which can be continued to the surface. If, in addition, the derivatives of order k belong to the class H-L with index α, we write $\varphi \in C^{k,\alpha}(\bar{D})$. Note that in this notation the belonging of a function to the class H-L is denoted as $\varphi \in C^{0,\alpha}(\bar{D})$. A surface S is said to be a surface of the class $\text{Л}_k(\alpha)$ if its equation $\xi_3 = \gamma(\xi_1, \xi_2)$ in a local co-ordinate system ξ_1, ξ_2, ξ_3 [the (ξ_1, ξ_2) plane is tangential and the ξ_3 axis is directed along the normal] is such that the function $\gamma(\xi_1, \xi_2)$ belongs to the class $C^{(k,\alpha)}$. The belonging of a surface to the class of Lyapunov surfaces is then denoted as $S \in \text{Л}_1(\alpha)$.

We introduce a generalized formulation of boundary value problems of the theory of elasticity. Suppose that the potentials constructed by means of the solutions of the integral equations (29.1), (29.3), and (29.6), with the above restrictions on the surface and the boundary conditions, give a solution of just generalized boundary value problems.

Below are given some results permitting an introduction of so severe restrictions on the surface and the boundary conditions that the solution of the integral equations leads to the construction of regular displacements (i.e., ensures the solution of the problem in the classical formulation).

(1) If $S \in \text{Л}_{n+1}(\alpha)$, $\varphi \in C^{n,\beta}(S)$ $(0 < \beta < \alpha)$, then the double layer potential $\mathbf{W}(\varphi) \in C^{n,\beta}(\bar{D})$.

(2) If $S \in \text{Л}_{n+1}(\alpha)$, $\varphi \in C^{n,\beta}(S)$ $(0 < \beta < \alpha)$, then the simple layer potential $V(\varphi) \in C^{n+1,\beta}(\bar{D})$.

(3) If the density of the simple layer potential $\varphi \in L_p$, then at all points of a surface $S \in \text{Л}_1(\alpha)$ there exist limiting boundary

* See T. G. Gegelia [1, 2] and also V. D. Kupradze, T. G. Gegelia *et al.* [1].

values (along a non-tangent path) defined by the formulas

$$W^{\pm}(\varphi) = \pm\varphi(p) + \int\limits_S \Gamma_2^1(p, q)\,\varphi(q)\,dS_q.$$

It will be recalled that in the case of a density $\varphi \in C^{0,\beta}$ we have the same formula, and the potential itself is continuously extendible.

Consider the properties of the derivatives (with respect to Cartesian co-ordinates) of a simple layer potential. If its density belongs to the class $C^{n,\beta}$, then there exist limiting values of the derivatives (along a non-tangent path), and in the case of $\varphi \in C^{0,\beta}$ they are continuously extendible to the surface.

We also give the following result. Let $S \in Л_{r+1}(\alpha)$, $f \in C^{r,\beta}$ $(\alpha > \beta > 0)$. Then every solution of the integral equations (29.1), (29.3), and (29.6) belonging to the class L_2 belongs to the class $C^{r,\beta}(S)$ as well.*

It appears from the above discussion that if $S \in Л_2(\alpha)$ and $f \in C^{1,\beta}$, then the solution of the integral equations leads to the solution of the boundary value problem in the classical formulation.

33. Approximate Methods of Solving Integral Equations for Fundamental Three-dimensional Problems

Numerical methods of solving integral equations are based in the first place on the possibility of evaluating the integrals appearing in the equations, no matter which method of solution used: whether it is the method of successive approximations (when the whole integrand is known at each stage) or the mechanical quadrature method (when the unknown function is assumed to be constant or varying in a certain manner within a small region, which enables one to pass to an integral of a known expression).

Let the surface S be divided into small (elementary) regions S_j $(j = 1, 2, \ldots, N)$. Denote by q_j the centre points, and by q_j^i the nodal points (the superscript indicates the numbering within each region S_i $(i = 1, 2, \ldots, N_j)$. It will be recalled that the points q_j are commonly termed pivotal, and the points q_j^i nodal, since the latter are naturally defined as the nodes of a curvilinear net.

Since the integrals appearing in Eqs. (29.1), (29.3), and (29.6) are singular, their evaluation at any point q_j requires, by difinition (see Sec. 7), such a discretization of the surface that the set of regions S_j adjacent to the point q_j is close to a region determined by its intersection with a circular cylinder of small radius whose axis passes through the point q_j along the normal to the surface. The

* We mention the work of A. A. Khvoles [1] where similar results have been obtained under less severe restrictions.

integral over the remainder of the surface gives an approximate value of the singular integral. Naturally, this approach is very laborious since it requires a different, at least local, discretization for each arrangement of points q_j (and there must be sufficiently many such points for a valid solution of an integral equation). True, if the small region in question is a plane polygon, the singular integrals under consideration can be evaluated in closed form (A. Ya. Aleksandrov [1], T. A. Cruse [1]). It is obvious that in the general case when the surface is curved, the application of cubature formulas virtually based on the polygonization of the surface entails a considerable loss of accuracy.

Another way of constructing cubatures for the singular integrals under consideration is to use the regular representations (31.17) and (31.18) reducing them to improper (regular) integrals. Such an approach opens up possibilities for constructing simple cubature formulas on a different basis.

As regards an estimate of the efficiency of particular cubature formulas (in reference to the solution of integral equations) the following must be kept in mind. A sufficiently high accuracy is of value only if it combines with ease of programming and little labour in debugging a computational scheme. The point is that the program debugging, as a rule, requires calculations for several discretizations (in order to establish the actual accuracy of solution), and it is therefore extremely desirable that the amount of work involved in altering the source information should be minimum.

We now turn to a direct consideration of the algorithms themselves. Let the integral equations be fulfilled at the pivotal points. The density is assumed to be constant within each region S_j and equal to its value at the pivotal point. The implementation of the mechanical quadrature method then reduces to a system of algebraic equations of order $3N$ ($9N^2$ coefficients). Note that in the case of the problem II^+ there may be difficulties since the system is close to a degenerate one (the determinants decrease with increasing order of the system). Attention is also drawn to the fact that the question of the convergence of approximate solutions to the exact one (with increasing N) is left open for the method under discussion since the corresponding proof is only available for the case of regular equations (see Sec. 11) and when the equation does not involve the eigenvalue.

We now consider the method of successive approximations in greater detail and point out some of its advantages. First, its implementation requires the retention of only the last iteration and the sum of the preceding ones (altogether $6N$ numbers) in the memory of a computer. Second, since there is a proof of convergence (in the case of its exact implementation) and the approximate determination of a finite sum of the series (see Sec. 10) reduces to a finite number of cubatures, it may be stated that the numerical algorithm leads to

a solution with given accuracy for a sufficiently fine discretization.

True, it may happen that the convergence of iterations in a particular problem is very slow and a direct implementation of the algorithm becomes more complicated. However, calculations show (and this agrees with the results following from the properties of the resolvent) that the functions $\varphi_n(q)$ begin fairly quickly to coincide with the terms of a geometric progression with a common (for all points of the surface) denominator (determined by the position of the second pole of the resolvent). As the denominator of the progression settles down to a stable (averaged) value, the ramainder of the series can therefore be summed up analytically.

Let us now consider the question concerning the solution of the integral equations for the problem II$^+$. As noted in Sec. 10, the implementation of an arbitrarily large number of iterations for a fixed discretization of the surface must lead to the divergence of the algorithm. If, however, the series is understood in the asymptotic sense, its convergence can be guaranteed. In this case the following procedure should be adopted. For a given number of terms of the series, perform calculations with decreasing size of the regions. This process necessarily converges since it reduces to evaluating a finite number of integrals. Further the number of terms of the series is increased, and the calculations are performed for a suitably chosen discretization. In the work of P. I. Perlin [10] a method is proposed in which a correction is made in each iteration in reference to Eq. (29.3):

$$\tilde{\varphi}_n(q') = \varphi_n(q') - \sum_{i=1}^{6} \psi_i(q') \int_S \psi_i(q)\, \psi_n(q)\, dS_q, \qquad (33.1)$$

where $\psi_i(q)$ is the orthonormal system of eigenfunctions of the companion equation (these functions are linear).

Note the following. Whatever the right-hand side of Eq. (29.3), series (10.9) must be convergent, and this means that

$$\lim_{n\to\infty} \varphi_n(q') = -\lim_{n\to\infty} \varphi_{n-1}(q') = \lim_{n\to\infty} \int_S \Gamma_1(q', q)\, \varphi_{n-1}(q)\, dS_q. \qquad (33.2)$$

Thus, the recurrence process (10.2) must lead to the construction of the eigenfunction of Eq. (29.3) with an arbitrary right-hand side.* Hence, though series (10.2) may be divergent, it will lead (with a sufficiently fine discretization) to stable stress values since the eigenfunctions determine only a rigid-body displacement.

Below are given some implementation schemes for the method of successive approximations on the basis of regular representations.

* Similar considerations in reference to an equation involving the eigenvalue $\lambda = 1$ are presented in the monograph of P. P. Zabreiko *et al.* [1].

The problem reduces to the following cubatures:

$$\begin{gathered}\int\limits_S \{\boldsymbol{\Gamma}_1(q', q)\,\varphi_{n-1}(q) - \boldsymbol{\Gamma}_2(q', q)\,\varphi_{n-1}(q')\}\,dS_q,\\ \int\limits_S \boldsymbol{\Gamma}_2(q', q)\,[\varphi_{n-1}(q) - \varphi_{n-1}(q')]\,dS_q.\end{gathered} \tag{33.3}$$

We first assume that the integrand is constant within each region. The integral sum at the point q_j is then represented as

$$\sum_{i=1}^{N}{}' \,[\boldsymbol{\Gamma}_1(q_j, q_i)\,\varphi_{n-1}(q_i) - \boldsymbol{\Gamma}_2(q_j, q_i)\,\varphi_{n-1}(q_j)]\,\Delta S_i, \tag{33.4}$$

where ΔS_i is the area and a prime indicates that the jth term is omitted. The remaining equations are obvious.

The approach leading to more exact cubature formulas is more difficult to realize. Let each term of the integral sum be represented as the product of the average value (over the nodal points) of the integrand and the area of the region. We then have

$$\sum_{i=1}^{N} \frac{1}{N_j}\left\{\sum_{k=1}^{N_j} [\boldsymbol{\Gamma}_1(q_j, q_i^k)\,\varphi_{n-1}(q_i^k) - \boldsymbol{\Gamma}_2(q_j, q_i^k)\,\varphi_{n-1}(q_j)]\right\}\Delta S_j. \tag{33.5}$$

As before, the calculation of the functions $\varphi_n(q)$ is carried out by numerical integration at the pivotal points. The calculation of these functions at the nodal points required for the implementation of formulas (33.5) is performed by interpolation between the nearest pivotal points. Naturally, it is advisable to rearrange the terms in (33.5) in each specific case.

Below are given the results of model calculations on the basis of the above algorithm. Consider the interior and exterior problems for a sphere. The load reduces to a unit hydrostatic pressure. The surface is divided by means of a geographic coordinate system (8 by 8). Table 1 gives values of the moduli of the functions $\varphi_n(q)$ ($n = 1, 2, 3, 4$), the corresponding sums, and the exact values* for two points of the surface: at the pole (point A) and at the equator (point B). The calculations were performed for $\nu = 0.3$. The discrepancy between the values at the points A and B is due to a difference in the actual size of the elementary regions.

* An example of constructing exact solutions for a spherical surface is discussed at the end of this section.

Table 1

Densities for Spherical Surface

	Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	Φ^-	Φ^+	Φ^-_{exact}	Φ^+_{exact}
A	1.000	0.247	0.063	0.015	0.004	—1.329	0.804	—1.312	0.807
B	1.000	0.269	0.064	0.016	0.004	—1.353	0.784	—1.312	0.807

Table 2 gives similar data for the case of a spheroidal surface. The surface was divided into 15 by 15 and 25 by 25 parts. The loading also reduced to a unit hydrostatic pressure.

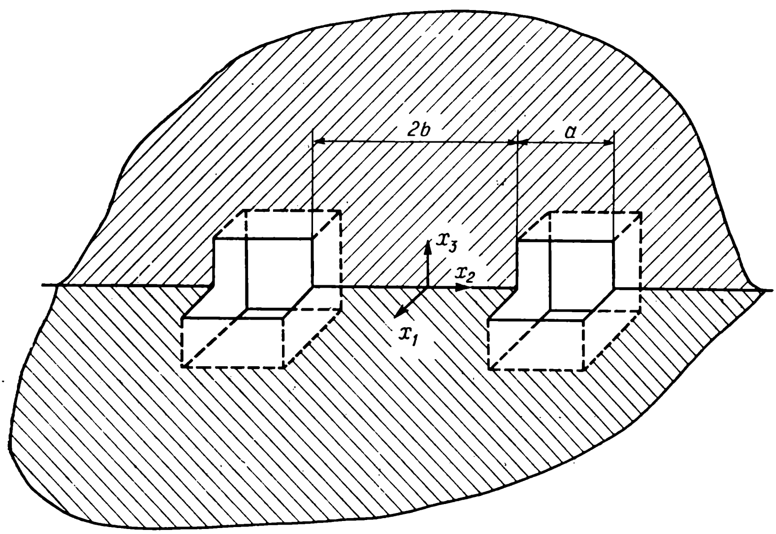

Fig. 16. Space with two cubic cavities

Let us now consider an example of practical interest (see E. M. Shafarenko [1]). Suppose that there are two identical cubic cavities in space (Fig. 16), which are subjected to a unit hydrostatic pressure.* The discretization of the surface is determined by a specification of several points at the edges. The straight lines joining them form a corresponding rectangular mesh. The calculations were carried out for 6, 8, and 10 points specified at the edges. Below are given the

* The questions concerning the irregularity of the boundary are considered in Sec. 37.

Table 2

Densities for Spheroidal Surface

		Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
15×15	A	1.0000	0.0389	0.1185	—0.0057	0.0155	—
	B	1.0000	0.3371	0.1065	0.036	0.0113	—
25×25	A	1.0000	0.0474	0.1208	—0.0034	0.0158	—0.0015
	B	1.0000	0.03367	0.1083	0.0367	0.0117	0.0040

Table 3

x_1	x_2	x_3	0	1	2	3	4	5
one cavity ($b=\infty$)								
0.5	0.1	0.1	1 (0)	0.382 (—0.053)	0.158 (—0.027)	0.078 (—0.01)	0.033 (—0.005)	0.015 (—0.002)
0.5	0.4	0.4	1 (0)	0.347 (—0.25)	0.345 (0.01)	0.088 (—0.02)	0.05 (—0.003)	0.018 (—0.003)
two cavities ($b/a=2$)								
0.1	1.5	0.1	0 (—1)	—0.053 (—0.377)	—0.027 (—0.157)	—0.01 (—0.077)	—0.005 (—0.032)	—0.002 (—0.015)
0.4	1.5	0.4	0 (—1)	—0.248 (—0.343)	0.01 (—0.342)	—0.02 (—0.087)	—0.003 (—0.048)	—0.002 (—0.017)
two cavities ($b/a=0.75$)								
0.1	0.25	0.1	0 (—1)	—0.035 (—0.24)	—0.015 (—0.085)	—0.033 (—0.048)	—0.033 (—0.013)	0 (—0.01)
0.4	0.25	0.4	0 (—1)	—0.192 (—0.265)	0.003 (—0.302)	0.003 (—0.04)	0.005 (—0.038)	0.007 (0.007)

results for eight points since the transition to ten points does not substantially affect the accuracy of solution. The subdivision of the edges into portions is not uniform: $0.05a$, $0.1a$, $0.15a$, $0.2a$ and in reverse order.

The calculations were carried out for the case of a single cavity ($b = \infty$), $b = 2a$ and $b = 0.75a$. Table 3 gives values of the components φ_n^1 and φ_n^2 (the latter in parentheses) at several points of the

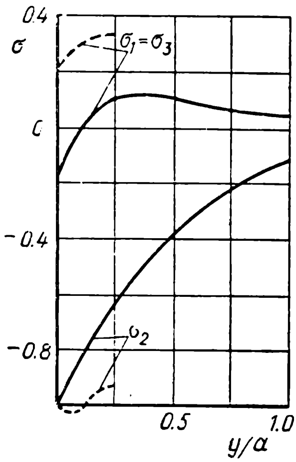

Fig. 17. Stresses σ_1, σ_2, σ_3 along a line joining the centres of cubic cavities

boundary whose co-ordinates are indicated. For the case of one cavity the origin is placed at its centre, and for two as shown in Fig. 16.

The number of iterations is seven. The error in fulfilling the integral equation does not exceed 0.5 per cent. Figure 17 gives the stress diagrams $\sigma_1 = \sigma_3$ and σ_2 along a line joining the centres of cavities, with partition thicknesses being different: $2b = 3a$ (solid lines) and $2b \doteq 0.5a$ (dashed lines). The calculations show that the solutions for one and two cavities with $b = 2a$ are practically the same. To check the accuracy for the given discretization, the solution of the interior problem is constructed. In this case the error for the stresses does not exceed 1 per cent.

It might be well to point out a procedure permitting a considerable reduction of the amount of calculations without any significant loss of accuracy (S. F. Stupak [1]). The calculation of $\varphi_n(q)$ is performed only at some of the pivotal points, and at the others the values are found by interpolation of one kind or another using the previously calculated values at the first group of points as the base. The questions of implementation of such an approach in reference to three-dimensional problems for bodies bounded by a set of surfaces frequently encountered in applications are studied in the work of E. M. Shafarenko and A. Z. Shternshis [1]. The authors also describe

another procedure of improving the efficiency of the algorithm. The idea is to use a mesh with a so-called variable interval introducing a local (fairly fine) discretization in the neighbourhood of a point q when calculating $\varphi_n(q)$ along with the global (general) discretization. The function $\varphi_n(q)$ is determined by interpolation at the resulting additional nodal points.

In the same work the authors suggest that a surface should be described by so-called shape functions widely used in the implementation of the finite element method (see O. C. Zienkiewicz, Y. K. Cheung [1]). The whole surface is divided into large portions each of which is mapped by means of a curvilinear coordinate system into a square in an auxiliary plane. Shape functions of the second order require eight points for their specification, namely vertices and middle points of the sides. After such a descretization the generation of a fine mesh required in the implementation of cubatures can easily be automated. To do this, it is necessary to specify several points on the sides of each of the squares. The square itself is then subdivided by segments joining these points into rectangles. The quadrilaterals formed on the original surface are curvilinear.

The division of a surface into portions is performed so that the approximation by functions of some order is sufficiently close to the surface. Moreover, it is necessary to allow for a different degree of discretization of the surface. It might be well to explain this. Suppose there is reason to think that a fine discretization is needed in a particular portion. If this portion is included in a large one (and not made into a separate one), it turns out that because of the construction adopted in automation an excessively fine (parasitic) mesh is generated in four adjacent regions, and this naturally reduces the total resource of the computational scheme.

Consider a special case of the three-dimensional problem for bodies of revolution (N. F. Andrianov [1]). In this case it is natural to perform discretization by parallels and meridians. Let a matrix $\mathbf{\Gamma}_1(q', q)$ be calculated for a pair of points q' and q. Before passing to a matrix $\mathbf{\Gamma}_1(q', q_1)$, where q_1 is, in general, an adjacent point to q, we construct a matrix $\mathbf{\Gamma}_1(q_1', q_1)$ for which the points q_1' and q_1 are obtained from q' and q by rotation through one division. It is easily seen that the elements of this matrix are the elements of the preceding matrix, which must therefore be retained in the memory of a computer. This procedure is continued until a complete revolution about the axis of rotation is performed. Of course, one may speak of the efficiency of the proposed techniques under loading conditions for which uniform (in angle) discretization is advisable.

Let us now study specific features of axially symmetric problems on the basis of the general equations for the three-dimensional problem. Naturally, in this case it is well to use parallels and meridians for discretization. Because of axial symmetry the vector $\varphi(q)$ and,

of course, each of the functions $\varphi_n(q)$ at the points of each parallel is constant in magnitude and lies in a meridional plane. The relations between its projections in a Cartesian co-ordinate system (this is the system used in integral equations) and a cylindrical co-ordinate system are of the form

$$\varphi_x = \varphi_r(r, z)\cos\varphi, \quad \varphi_y = \varphi_r(r, z)\sin\varphi. \tag{33.6}$$

Let the order of evaluation of the integral sum be such that meshes on one parallel are passed in succession. It is then possible (S. F. Stupak [1]) to sum numerically the corresponding combinations of integrands having the factors $\cos\varphi$ and $\sin\varphi$. As a result, all the information required to perform iterations is so compact that it can be entered into the storage of a computer. Hence, the calculation of each iteration (after this preparatory work) does not take much time. Later, in the work of V. G. Kostylev and N. F. Andrianov [1] the integration with respect to the angle was performed explicitly, which led to elliptic integrals.

Let us consider procedures for solving integral equations by the mechanical quadrature method. Based on his cubature formulas, T. A. Cruse [1] has obtained solutions of several model problems (the exterior and interior problems for a sphere) using Eqs. (29.6). Since the above cubatures require polygonization of the surface, a discretization of the surface (unsuitable, in the author's opinion) has been made leading to a significant error.

A. Ya Aleksandrov [7] constructs a solution of Eq. (29.3). The procedure of constructing the cubatures is such that it, in fact, immediately leads to the calculation of the left-hand side of the equation. The author assumes a load uniformly distributed over a rectangle. By summing the Kelvin-Somigliana solutions, it is possible to determine the stresses from the two sides of the area. By symmetry, these stresses are numerically equal, but, according to (27.13), their difference is equal to a given load. The limiting values are therefore equal to half the load. Note that approximate formulas are proposed for calculating the regular terms. The resulting system of algebraic equations is solved by successive approximations. In the same work a specific example is given: the solution of the problem for a cube subjected on two opposite faces to normal loads uniformly distributed over a square. The lack of an exact solution or a solution for several discretizations prevents an error estimation.

In the work of J. C. Lachat, J. O. Watson [2] it is noted that the use of cubature formulas based on the surface requires a very fine discretization to attain a reasonable accuracy. Taking into account the variation of the density in each polygon did not lead to any significant improvement of the algorithm. On this account, apparently, it has been suggested that the cubature formulas should be

constructed, in fact, on the basis of the same formula (31.17) using generalized Gauss' theorem (27.8) (T. A. Cruse [4]).

Note that in the case where the boundary surface is a sphere the solution of the integral equations can be obtained as series in associated Legendre polynomials. The construction of the solution calls for the complete set of particular solutions of the elasticity problem for the outer and inner regions (see A. I. Lur'e [1]). By considering the corresponding pair of solutions (for the inner and outer regions) as a simple layer potential (the displacements must be identical on the surface), its density can be determined by using formula (28.13). By performing the inverse procedure, the potential can be restored. The boundary condition is therefore expanded in a series of polynomials, and a solution is found for each harmonic. This procedure has been used in the works of P. I. Perlin [4] and P. I. Perlin and S. F. Stupak [1] in a certain stage of the solution of axially symmetric problems for bodies bounded by two surfaces.

D. G. Natroshvili [1] was able to sum up the solution in series, which led to the construction of the Green's function for a sphere.

The foregoing will be illustrated by a simple example. Let the problem be solved for a sphere of radius R subjected to a hydrostatic pressure p^+. In spherical co-ordinates we have

$$U_r = \frac{p^+ r}{3\lambda + 2\mu}, \qquad \sigma_r \equiv p^+. \tag{33.7}$$

The corresponding simple layer potential in the outer region obviously leads to a solution depending only on the radius. Let p^- be the corresponding limiting value of the stress vector from the outside of the region. The solution itself is then represented as

$$U_r = -\frac{p^- R^3}{4\mu r^2}, \qquad \sigma_r = \frac{p^- R^3}{r^3}. \tag{33.8}$$

By equating the displacements on the surface, we obtain

$$p^- = -p^+ \frac{3\lambda + 2\mu}{4\mu}.$$

The jump in the stresses is equal to twice the density:

$$2\varphi = p^+ - p^- = p^+ \frac{3(1-\nu)}{1+\nu}. \tag{33.9}$$

The results of calculation by this formula for $\nu = 0.3$ have been previously given in Table 1.

After the solution of the integral equations is obtained in some way or other, the stresses and displacements must be calculated. For the case of interior points this problem involves no difficulties of principle since it reduces to that of evaluating the corresponding integrals of regular kernels obtained by differentiating the matrix

$\mathbf{\Gamma}(q', q)$ and possibly $\mathbf{\Gamma}_2(q', q)$. It should be noted that the stress tensor can be obtained directly from the integral

$$\int_S \mathbf{\Gamma}_1(p, q)\,\varphi(q)\,dS_q \tag{33.10}$$

by assigning successively three normal directions at the point p parallel to the co-ordinate axes.

It should be remembered (especially in calculations at points near the boundary) that it is necessary to introduce an additional discretization (along with the original one used in solving the integral equation). Of course, this applies to the part of the surface that is close to the points under consideration.

The problem of determining the stresses at points situated on the surface is more complicated. If the polygonization of the surface has been performed (it now suffices to perform this procedure in the neighbourhood of the point under investigation), one can use the formulas resulting from the constructions of A. Ya. Aleksandrov [7] and T. A. Cruse [1].

In the general case the regular representations (31.18) can only be applied for the stress components that enter into the boundary conditions (it is worth-while to use them as a check on the accuracy). The stress components that do not enter into the boundary conditions can be determined by a procedure originally suggested by P. I. Perlin and V. N. Samarov [1] for somewhat different purposes (see Sec. 35). Its justification follows from the differential properties of a simple layer potential whose density is the solution of the integral equation (see Sec. 32). The displacements (unlike the stresses) are calculated rather easily on the boundary since here we have to deal with improper integrals. Obviously, after calculating the displacements at the points of the boundary, we can find three components of the strain tensor. By using three components of the stress tensor known from the boundary conditions, we can determine the remaining ones by Hooke's law.

In solving the problem I (by means of the double layer potential), it is well to determine the displacements at points located near the surface as follows. Let a point q' be chosen on the surface in the immediate vicinity of the point p at which the displacements are calculated. Next the identity following from Gauss' theorem (28.8) is used. The result is

$$U(p) = \int_S \mathbf{\Gamma}_2(p, q)\,[\varphi(q) - \varphi(q')]\,dS_q - \alpha\varphi(q') \tag{33.11}$$

where $\alpha = 0$ for the exterior problem, and $\alpha = 2$ for the interior problem.

Suppose, now, that the second fundamental problem has been solved using Eq. (29.6). The displacements on the boundary are then

determined directly from the integral equation. The stresses are therefore most easily found by differentiating the displacements. It should be noted that, in general, the determination of the stresses in this case at points close to the boundary is a more complicated task than in the case of Eq. (29.3) since the double layer potential has to be differentiated as well. Similar difficulties arise when the problem I is solved with the help of Eq. (29.1).

As has already been noted, the solution of the second fundamental problem can be carried out using either Eq. (29.3) or Eq. (29.6). In principle they are completely equivalent, being companion to each other. Let us compare them from the standpoint of computational implementation. An apparent disadvantage of Eq. (29.6) is that the right-hand side has to be calculated beforehand. Since it is an improper integral, the calculations will obviously involve a significant error. Of course, the loading conditions may be such that the forces are zero over part of the surface, and the evaluation of integrals is therefore simplified. A certain advantage in that the displacements on the surface are found directly from the solution of the integral equation does not appear to be so important. The point is that in using Eq. (29.3) to determine displacements, the same improper integrals must be evaluated, but generally only at several boundary points. There are also difficulties in determining the stresses at points near the boundary (as has already been noted).

Thus, in our opinion Eq. (29.3) is more preferable.*

We now turn to the consideration of numerical procedures for solving three-dimensional problems, also based on potential methods. V. D. Kupradze [4] has proposed a method for the solution of the fundamental three-dimensional problems of the theory of elasticity using functional equations obtained from Betti's formulas:

$$\int_S \mathbf{\Gamma}_2(p,\ q)\,\boldsymbol{U}(q)\,dS_q - \int_S \mathbf{\Gamma}(p,\ q)\,\boldsymbol{T}_n\boldsymbol{U}(q)\,dS_q = 0 \qquad (14.30')$$

for all points p not belonging to the region in which the solution of the boundary value problem is sought. In the case of the first fundamental problem the first term is prescribed, and in the case of the second fundamental problem the second. Accordingly, the unknown functions are the external stresses or the displacements on the boundary. Several points are chosen outside the region. It is required that Eq. (14.30′) should be fulfilled at these points. Next the surface S is discretized into the same number of portions, and the unknown functions are assumed to be constant within each portion. By evaluating the corresponding integrals, we arrive at a system of algebraic equations. The questions of actual implementation of this algorithm are discussed in the monograph of Yu. V. Veryuzhskii [1].

* Note that American authors construct the solution using Eq. (29.6).

In the work of N. M. Khutoryanskii [1] a system of functional relations is constructed by expanding the kernels of the functional relation (14.30′). It is shown that this system falls into a set of independent systems for bodies of revolution.

The functional equations under consideration serve as the basis for a method called the generalized series method (V. D. Kupradze and M. A. Aleksidze [1]). To implement it, a surface is taken outside the body and a countable set of points is chosen on it. Particular solutions (corresponding to concentrated forces applied at these points) are used to construct a set of orthonormal, average complete functions on S. It can be shown that explicit expressions in integral form are obtained for the Fourier coefficients of the unknown function.

In conclusion we discuss the solution of plane elasticity problems. Note that in contrast to the three-dimensional case the solution of the equations for the plane problem reducing to one-dimensional singular integral equations involves no difficulties of principle. As noted in Sec. 12, the evaluation of one-dimensional singular integrals can be performed directly by formulas of sufficiently simple structure. The specific features of the singular integrals appearing in Eqs. (29.19) and (29.20) make it possible to use regular representations.

In the work of F. J. Rizzo [1] a numerical method is proposed for solving the integral equation (29.20).* The contour of integration is divided into a number of arcs by points t_k, and a point is chosen on each of them denoted by t_k^0 and equidistant from t_k and t_{k+1}. The displacements $U(t)$ are assumed to be constant within each arc and are assigned to the appropriate point t_k^0. The solution of the integral equation is performed by equating the left-hand and right-hand sides at all points t_k^0. The necessary values of the singular terms are found in explicit (elementary) form since the arc t_k, t_{k+1} is replaced by a broken line t_k, t_k^0, t_{k+1} and actually the angle between t_k, t_k^0 and t_k^0, t_{k+1} is calculated. The regular terms are found by Simpson's formula.

In Rizzo's work some calculations are given. The author considers a problem for a circle loaded by a normal pressure whose relation to the polar angle is of the form $N = P\cos^2\varphi$. The solution is constructed for 12 and 24 arcs of equal length. The position and numbering of portions are chosen so that the point t_1^0 corresponds to the angle $\varphi = 0$. Table 4 gives values of the relative error in displacements Δ_1 and Δ_2 (as fractions of the maximum displacement)** at points of one quadrant. Poisson's ratio is $\nu = 0.25$.

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* Note that a method for solving the singular integral equations of N. S. Kakhniashvili [1] has been previously proposed by A. Ya. Aleksandrov [1].

** The projections of displacement on the directions $\varphi = 0$ and $\varphi = \pi/2$ are meant. The error is determined by the exact solution.

Table 4

Error in Displacements for a Circle with Two Divisions

Points	$n = 12$		Points	$n = 24$	
	Δ_1	Δ_2		Δ_1	Δ_2
1	0.0482	0.0000	1	0.0268	0.0000
			2	0.0185	0.0211
2	0.0003	0.0601	3	0.0000	0.0313
			4	0.0184	0.0258
3	0.0519	0.0165	5	0.0261	0.0091
			6	0.0184	0.0085
4	0.0000	0.0317	7	0.0000	0.0157

A solution is also obtained of the problem for a rectangle with a side ratio of 8. The loading reduces to a pressure uniformly distributed along the shorter sides. The contour is divided into 48 parts so that the centre points are off the corners and the portions are of the same length. The numbering is chosen so that the point t_1^0 corresponds to the middle of the shorter side. Table 5 gives errors in displacements (Δ_1 and Δ_2) at points of one quadrant. Poisson's ratio is $\nu = 0.2$.

Table 5

Error in Displacements for a Rectangle

Points	Δ_1	Δ_2	Points	Δ_1	Δ_2
1	0.0036	0.0000	5	0.0033	0.0138
2	0.0026	0.0015	6	0.0013	0.0126
3	0.0050	0.0058	7	0.0005	0.0118
4	0.0102	0.0138	8	0.0005	0.0115

The questions of the efficient solution of the singular equations (29.21) by the mechanical quadrature method using interpolation polynomials are studied in the work of Yu. D. Kopeikin, M. I. Alyautdinov and Yu. L. Bormot [1].

Consider the specific features of the solution of integral equations in elasticity when Poisson's ratio is close to 0.5. If we set $\nu = 0.5$ in the integral equations, the equations obtained are identical with those for a linearized flow of viscous incompressible fluid derived and investigated by F. K. G. Odqvist [1]. It is shown that these equa-

tions have the same spectral properties as Eqs. (29.1) and (29.3) and, in addition, the point $\lambda = 1$ is a simple pole of the resolvent. The eigenfunction for the equation of the second exterior problem is known; this is a vector function directed along the normal to the surface and having a constant length $\varphi^*(q) = Cn$.

It should be borne in mind that a direct study of elasticity problems for an incompressible medium cannot be adequately made on the basis of the integral equations (29.1) and (29.3) if $\nu = 0.5$ is substituted in them, but it seems advisable to use the properties of these equations to study the behaviour of the solution for values of Poisson's ratio close to 0.5.

Since the point $\lambda = 1$ is a pole of the resolvent for an incompressible medium, it is natural to expect that as Poisson's ratio tends to 0.5, the convergence of the foregoing algorithms (based on successive approximations) becomes worse (since the second pole of the resolvent tends to unity). There is no difficulty in obtaining a convergent representation in the case of the problem II^+. It is sufficient to use the method of elimination of the pole of the resolvent by remultiplication (see Sec. 10). For $\nu = 0.5$, the following series is then convergent:

$$\varphi(q) = 0.5\varphi_0(q) + 0.5\,[\varphi_0(q) - \varphi_1(q)] - 0.5\,[\varphi_1(q) - \varphi_2(q)] + \\ + 0.5\,[\varphi_2(q) - \varphi_3(q)] - \ldots . \qquad (33.12)$$

Naturally, the series is also convergent for values of ν close to 0.5. It is therefore apparent that the construction of the solution on the basis of (33.12) requires for its implementation a much more coarse discretization than would be the case if one proceeded directly from series (10.2).

In the work of P. I. Perlin, A. V. Novikov and S. F. Stupak [1] a procedure is suggested for the solution of the integral equation for the problem II^- when $\nu = 0.5$. At the first stage the integral equation is solved by successive approximations; the number of iterations is established from the following considerations. Since series (33.12) is convergent, we have the equality

$$\lim_{n\to\infty} \varphi_{n-1}(q') = \lim_{n\to\infty} \varphi_n(q') = \lim_{n\to\infty} \int\limits_S \Gamma_1(q', q)\,\varphi_{n-1}(q)\,dS_q. \qquad (33.13)$$

Consequently, the recurrence process must lead to the eigenfunction, which is known. With a fairly fine discretization and an appropriate number of iterations, we therefore arrive at the modulus of the eigenfunction (C). Further the equation is solved by successive approximations with boundary conditions for which the solution of the problem in terms of stresses is known. In this case we also arrive at a certain value of the constant C_1. At the last stage the problem is considered with a boundary condition obtained from the original

one by subtracting the condition of the auxiliary problem multiplied by the ratio C/C_1. It is obvious that the algorithm will now converge.

In solving the problem II^- for a medium approximating an incompressible one, it is therefore recommended that the boundary conditions should be transformed as indicated (i.e., using the equations for $\nu = 0.5$). The algorithm will then converge for a more coarse discretization than in the direct solution of the problem.

34. Problems of the Theory of Elasticity for Bodies Bounded by Several Surfaces

Let the region D be bounded by several surfaces, $S_0, S_1, \ldots, S_m$, and let the surface S_0 (which may be absent) enclose all the rest lying outside each other. It is required to solve the elasticity problem for the region D when certain conditions are prescribed over the surfaces S_j.

If displacements are prescribed over all surfaces, the solution is sought in the form of a double layer potential of the first or second kind, and if stresses are prescribed, the solution is sought in the form of a simple layer potential. We arrive at integral equations that may be symbolically represented in the same form as Eqs. (29.1) to (29.3) if by S is meant a union of all surfaces and the equations are considered on each surface. Thus, a system of integral equations is, in fact, constructed for all densities $\varphi_j(q)$ (the subscript of the function corresponds to that of the surface). The parameter ν is equal to 1 in the problem I, and to -1 in the problem II.

Note that Betti's formulas proved in Sec. 16 are automatically generalized to the regions under consideration, which leads to the extension of some results pertaining to the spectral properties of these equations (see Sec. 31) (all eigenvalues are real and not less than unity in magnitude). The applicability of the Fredholm alternatives in the case of singular equations is also obvious since the additional operators are totally continuous. The main difficulty is due to a more complex structure of the resolvent in the neighbourhood of $\nu = \pm 1$.

To carry out the corresponding analysis of the equations, which will be further referred to as (29.1) to (29.3), we introduce into consideration an auxiliary (companion) problem, a problem for a region D' composed of the regions $D_0^-, D_1^+, \ldots, D_m^+$. Though this problem can be divided into a number of independent elementary boundary value problems, its consideration by assigning the densities of certain potentials on all surfaces simultaneously is of interest since it leads to companion integral equations.

Let displacements be prescribed over the surfaces S_j. Then there exist non-trivial solutions of the homogeneous problems both for

Eq. (29.1) and Eq. (29.2). They are of the form

$$\varphi_0(q)=0, \quad \varphi_j(q)=C_j \quad (j=1, 2, \ldots, m) \tag{34.1}$$

where C_j are constant vectors.

From the uniqueness theorems it follows that there can be no other (linearly independent) solutions. Thus, Eqs. (29.1) and (29.2) are in general unsolvable.

To construct solvable regular equations, D. I. Sherman [10] suggested a modification of the representation of displacements, which develops the procedure previously devised by him for studying the Dirichlet problem (see D. I. Sherman [8]). The author introduced the following terms into the representation for displacements:

$$\mathbf{\Gamma}(p, p_j)\int\limits_{S_j}\varphi_j(q)\,dS_q, \tag{34.2}$$

where the points p_j are chosen arbitrarily in the regions D_j^+. The equation uniquely solvable is of the form

$$\varphi_i(q)-\nu\sum_{j=0}^{m}\int\limits_{S_j}\mathbf{\Gamma}_2^{\text{II}}(q, q')\varphi_j(q')\,dS_{q'}+\sum_{j=1}^{m}\mathbf{\Gamma}(q, p_j)\int\limits_{S_j}\varphi_j(q')\,dS_{q'}=0$$

$$(q\in S_i) \quad (i=0, 1, 2, \ldots, m). \tag{34.3}$$

It follows from the uniqueness theorem that the solution of (34.3) [denote it by $\varphi^*(q)$] satisfies the homogeneous functional equation, which should be written as

$$\nu\int\limits_{S_0}\mathbf{\Gamma}_2^{\text{II}}(p, q)\varphi_0^*(q)\,dS_q=\nu\sum_{j=1}^{m}\int\limits_{S_j}\mathbf{\Gamma}_2^{\text{II}}(p, q)\varphi_j^*(q)\,dS_q+$$

$$+\sum_{j=1}^{m}\mathbf{\Gamma}_2(p, p_j)\int\limits_{S_j}\varphi_j^*(q)\,dS_q \quad (p\in D). \tag{34.4}$$

The positive sense of the normal in Eqs. (34.4) is chosen out of the body. Let the expression on the left-hand side be denoted by $U_1(p)$, and the expression on the right-hand side by $U_2(p)$. Either of them may also be considered in the region D_0^-. The displacements $U_1(p)$ have a discontinuity on the surface S_0, and the limiting values of the N-operator coincide. The surface S_0 is not a singular surface for the displacements $U_2(p)$. Since the limiting (from the inside) values of the N-operator of $U_1(p)$ and $U_2(p)$ coincide, according to (34.4), it follows that the limiting values from the outside also coincide, and this leads to coincidence of the displacements $U_1(p)$ and $U_2(p)$ in the region D_0^-. From the continuity of the displacements $U_1(p)$ across S_0 it follows that the function $\varphi_0^*(q)$ is zero.

Taking this into account, we rewrite the functional equation (34.4) for an arbitrarily chosen subscript i as

$$\nu \int_{S_i} \boldsymbol{\Gamma}_2^{\mathrm{II}}(p, q)\, \varphi_i^*(q)\, dS_q + \boldsymbol{\Gamma}(p, p_i) \int_{S_i} \varphi_i^*(q)\, dS_q =$$

$$= -\nu \sum_{j=1,\ j\neq i}^{m}{}' \int_{S_j} \boldsymbol{\Gamma}_2^{\mathrm{II}}(p, q)\, \varphi_j^*(q)\, dS_q -$$

$$- \sum_{j=1,\ j\neq i}^{m}{}' \boldsymbol{\Gamma}_2(p, p_j) \int_{S_j} \varphi_j^*(q)\, dS_q \quad (p \in D). \qquad (34.5)$$

The expression on the left-hand side may be considered in the whole region D_i^-, and the expression on the right-hand side in the regions D and D_i^+. Since these expressions coincide in the region D, they can be continued into each other and represent a unique function satisfying Lamé's equations in the entire space and therefore equal to zero. We rewrite the equality proved above:

$$\int_{S_i} \boldsymbol{\Gamma}_2^{\mathrm{II}}(p, q)\, \varphi_i^*(q)\, dS_q + \boldsymbol{\Gamma}(p, p_i) \int_{S_i} \varphi_i^*(q)\, dS_q = 0 \quad (p \in D_i^-).$$

The first term decreases at infinity as R^{-2}, and the second as R^{-1}. Both terms are therefore zero, i.e.,

$$\int_{S_i} \boldsymbol{\Gamma}_2^{\mathrm{II}}(p, q)\, \varphi_i^*(q)\, dS_q = \int_{S_i} \varphi_i^*(q)\, dS_q = 0 \quad (p \in D_i^-). \qquad (34.6)$$

Consider now the potential $\int_{S_i} \boldsymbol{\Gamma}_2^{\mathrm{II}}(p, q)\, \varphi_i^*(q)\, dS_q$ in the region D_i^+. From the continuity condition for the N-operator it follows that this potential can take only a constant value, which must be zero by (34.6). It appears from the above discussion that the function $\varphi_i^*(q)$ vanishes. Because of the arbitrariness in the choice of subscript this proves that the function $\varphi^*(q)$ is identically zero. Consequently, the integral equation (34.3) is always solvable.

Let us consider the solution of the problem I with the aid of the singular integral equation (29.2). In the work of T. V. Burchuladze [1], which is a generalization of the approach to the solution of the Dirichlet problem developed by V. D. Kupradze [1], it is suggested that the displacements should be sought in the form

$$\boldsymbol{U}(p) = \sum_{j=0}^{m} \int_{S_j} \{\boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q) - \boldsymbol{\Gamma}(p, q)\}\, \varphi_j(q)\, dS_q, \qquad (34.7)$$

this leads to the singular integral equation

$$\varphi_i(q) - \nu \sum_{j=0}^{m} \int_{S_j} \{\Gamma_2^{\mathrm{I}}(p, q') - \Gamma(q, q')\} \varphi_j(q')\, dS_{q'} = F(q)$$
$$(q \in S_i) \quad (i = 1, 2, \ldots, m). \qquad (34.8)$$

This equation differs from Eq. (29.2) only by the regular term, and hence the Fredholm alternatives remain applicable.

To analyze Eq. (34.8), we now turn to the so-called *fifth* (in the terminology of the work of V. D. Kupradze and T. G. Gegelia [1]) *fundamental problem of the theory of elasticity* when the following relation is prescribed on the bounding surfaces:

$$T_n U(q) + \sigma(q) U(q) = f(q), \qquad (34.9)$$

where $\sigma = \| \sigma_{kj} \|$ $(k, j = 1, 2, 3)$, and the form $\sum \sigma_{kj} x_k y_j$ is positively definite.

The solution of this problem is sought for the region D' by representing the displacements as a simple layer potential. In the case of the homogeneous boundary conditions we arrive at a singular equation companion to (34.8). Let $\psi^*(q)$ be its non-trivial solution and let $V(p, \psi^*)$ be the corresponding potential. From the uniqueness theorem for the fifth fundamental problem it follows that this potential must vanish in the regions D_0^- and D_j^+. From the continuity of the simple layer potential we conclude that it is zero in the region D, and this proves the absence of non-trivial solutions for the homogeneous companion equation. Thus, Eq. (34.8) is always solvable.

As a special case, the above approaches enable us to consider the problem I^- (when there is a single surface S_1).

It should be noted that the question of the convergence of the method of successive approximations for the modifications considered above is left open since we virtually deal with new integral equations in general.

In the work of N. I. Muskhelishvili [2] a method is proposed for constructing integral equations for the Dirichlet problem, which can easily be extended to elasticity problems. Instead of the original problem, a modified problem is considered when the displacements on the boundaries are determined to within an arbitrary rigid displacement of each surface S_j. The representations of displacements are supplemented by terms of the form $\Gamma(p, p_k) C_k$, with the constant vectors C_k to be determined. The additional integral operators introduced into the resulting equations are such that their kernels vanish when the points p and q belong to different surfaces. When the arguments are on the same surface, the kernels represent a constant matrix. The integral equations thus obtained are always solvable. At the final stage the constants C_k must be determined so that the

modified problem coincides with the original one; this leads to a system of algebraic equations that is always solvable. The structure of the added terms is such that every solution of the modified equation is the solution of the original equation. In the same work of N. I. Muskhelishvili [2] it is shown that the eigenvalues are greater than unity in magnitude, and in consequence the equations can be solved by the method of successive approximations.

We now turn to the consideration of the second fundamental problem. The homogeneous equation companion to Eq. (34.3) is an integral equation, which can be obtained by solving the first fundamental problem simultaneously for the regions of the system D' proceeding from the representations of displacements as a double layer potential of the first kind with zero displacements on the boundaries. Obviously, the displacements themselves are zero, and it follows from the continuity of the stress operator that the displacements in the region D generated by these potentials can correspond to a rigid-body displacement. Consequently, a non-trivial solution of the companion equation exists and it is elementary. It can also be shown that no other non-trivial solutions exist (see T. V. Burchuladze [1]). Thus, the integral equations (34.3) are solvable if the orthogonality conditions are fulfilled, which, as in the problem II^+, express the fact that the resultant vector and the resultant moment of the external forces are zero. Note that in the absence of the surface S_0 the problem is always solvable.

Equation (34.3) can be solved by the method of successive approximations in a modified form [for example, (10.7)] by reexpanding in a power series in the ν plane around a point situated between $\nu = 0$ and $\nu = -1$. To answer the question of the possibility of direct solution [i.e., in the form of series (10.1)], it is necessary to examine Eq. (34.3) when $\nu = 1$, i.e., when solving the problems for the regions of the system D'. It is obvious that the integral equations corresponding to these problems have eigenfunctions, which are found in an elementary way since they correspond to an independent rigid-body displacement of the surfaces S_j $(j = 1, \ldots, m)$. To eliminate the poles of the resolvent at the point $\nu = 1$, it is necessary to require the fulfilment of the orthogonality conditions for each of these eigenfunctions, which is equivalent to the condition that the resultant vector and the resultant moment of the external forces applied to each surface should be zero. This restriction on the generality of boundary conditions is purely apparent since a simple transformation leads to their fulfilment (by superimposing the solution for concentrated forces and moments applied at points of the regions D_j^+).

A generalization of the approach to the solution of the Neumann problem proposed in the work of N. I. Muskhelishvili [2] to include the theory of elasticity leads to similar conclusions concerning the pos-

sibility of solving Eqs. (34.3) directly by the method of successive approximations. The integral equations suitably modified can also be solved by the method of successive approximations if the conditions on the load specified above are fulfilled.

Naturally, the solution of three-dimensional problems of elasticity in the presence of several bounding surfaces can be carried out in the same way as for a single surface (using regular representations). Of course, the computer resources must be greater in this case. The lack of calculations using this kind of procedure prevents an assessment of the efficiency of these approaches at present. It might be expected that if the (local or global) distance between the surfaces is decreased, it will be necessary not only to increase the number of iterations, but also to use a progressively finer discretization of the surface.

The construction of integral equations when boundary conditions of different kind are prescribed on the surfaces S_j is obvious.

Let us now consider a method of solving problems for bodies bounded by two surfaces which has been proposed by P. I. Perlin [2, 5] as a generalization to the three-dimensional case of the method due to D. I. Sherman [12]. It is required to solve a problem when the region is bounded by surfaces S_0 and S_1 over which stresses $\boldsymbol{f}_0(q)$ and $\boldsymbol{f}_1(q)$ are prescribed. For one of the additional regions, say D_0^-, we then solve a problem with the same boundary condition $\boldsymbol{f}_0(q)$. Let the resulting displacements on the surface S_0 be denoted by $\boldsymbol{\lambda}_0(q)$. Next, an auxiliary function $\boldsymbol{\lambda}(q)$ is introduced on this surface, equal to the required displacement. It can be shown that the displacement $\boldsymbol{U}_1(p) = \boldsymbol{U}(p) + \boldsymbol{W}(p, \boldsymbol{\mu})$, where $\boldsymbol{\mu}(q) = 0{,}5\,[\boldsymbol{\lambda}(q) - \boldsymbol{\lambda}_0(q)]$, satisfy Lamé's equations in the region D_1^-. We solve this problem (with an assumed function $\boldsymbol{\mu}(q)$ and determine displacements on the surface S_0, by equating which to the function $\boldsymbol{\lambda}(q)$ we arrive at a singular integral equation for it.

If displacements or displacements and stresses are prescribed on the bounding surfaces, the procedure remains virtually the same.

In the works of P. I. Perlin [4] and P. I. Perlin and S. F. Stupak [1] solutions have been obtained for the first and second problems for regions bounded by an ellipsoid and a sphere. All constructions have been performed by using the series technique.

35. Three-dimensional Problems of the Theory of Elasticity for Bodies with Cuts

In the preceding sections of this chapter we have considered methods of solving three-dimensional problems of elasticity for bodies bounded by smooth surfaces. We now turn to the study of a special case when there is a cut in the form of an unclosed Lyapunov surface S in a finite or an infinite elastic body. This surface must, in gener-

al, be considered two-sided denoting one of its sides by S^+ and the other by S^-, whose points are, respectively, q^+ and q^- (the positive direction of the normal is chosen towards S^-). The smooth contour bounding the surface S is denoted by L.

Below is given the formulation of the second fundamental problem. It is required to determine, in the whole space, an elastic displacement vector $\boldsymbol{U}(p)$ satisfying the limiting conditions

$$\lim_{p\to q^+} \boldsymbol{T}_n\boldsymbol{U}(p) = \boldsymbol{f}^+(q^+), \quad \lim_{p\to q^-} \boldsymbol{T}_n\boldsymbol{U}(p) = \boldsymbol{f}^-(q^-), \tag{35.1}$$

assuming that the functions $\boldsymbol{f}^+(q^+)$ and $\boldsymbol{f}^-(q^-)$ are given and belong to the class H-L. Assume also that the stresses vanish at infinity. If, however, the stresses at infinity are different from zero, it is easy to come to the case under consideration by superimposing a trivial solution for the continuous body.

We form a simple layer potential $\boldsymbol{V}(p, \boldsymbol{f}_1)$ with density $\boldsymbol{f}_1(q) = 0.5\,[\boldsymbol{f}^+(q^+) - \boldsymbol{f}^-(q^-)]$. It can be shown that the displacement $\boldsymbol{U}_1(p) = \boldsymbol{U}(p) - \boldsymbol{V}(p, \boldsymbol{f}_1)$ satisfies the same (stress) boundary conditions on the surface S. Suppose that this transformation has already been made and solve the boundary value problem when the functions $\boldsymbol{f}^+(q^+)$ and $\boldsymbol{f}^-(q^-)$ coincide, omitting the superscript in conditions (35.1).

The displacement is now sought in the form of a double layer potential of the first kind distributed over the surface S:*

$$\boldsymbol{U}_1(p) = \boldsymbol{W}(p) = \int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q)\,\boldsymbol{\varphi}(q)\,dS_q. \tag{35.2}$$

As is known, the potential $\boldsymbol{W}(p)$ satisfies Lamé's equations in the whole elastic body, and the limiting stress values generated by it from the two sides of the surface coincide** (see Sec. 28). Thus, the solution of the elasticity problem reduces to the solution of the functional equation

$$\lim_{p\to q^\pm} \boldsymbol{T}_n \int_S \boldsymbol{\Gamma}_2^{\mathrm{I}}(p, q)\,\boldsymbol{\varphi}(q)\,dS_q = \boldsymbol{f}(q^\pm). \tag{35.3}$$

Note that the unknown density $\boldsymbol{\varphi}(q)$ is equal to half the jump in the displacements on the surface S.

The discretization of the surface S is performed by dividing it into small triangles S_j ($j = 1, 2, \ldots, N$). We choose, in each region S_j, a point located in the central part, which (according to the terminology introduced in Sec. 33) will be referred to as pivotal and denoted by p_j. Assume the density to be constant (within each small

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* This procedure is a generalization of the method proposed by F. G. Tricomi [1] for the solution of similar harmonic problems.

** Assuming that these limiting values exist.

region S_j) and equal to its value at the point p_j ($\varphi(p_j) = \varphi_j$). Let us now calculate the left-hand side of Eq. (35.3) at each pivotal point. The corresponding integral sum must be of the form

$$\sum_{k=1}^{N} \alpha_{jk}\varphi_k.$$

Each term of this sum is the product of the third-order matrix α_{jk} and the vector φ_k. In practice, this sum must be constructed as follows. At the point p_j we assign a vector φ_j^1 having the first component of unit magnitude and the remaining two equal to zero. All components of the vector φ at all other points are assumed to be zero.

After corresponding calculations (to be discussed below) we find the first rows of the matrices α_{jk} ($k = 1, 2, \ldots, N$). Assuming next only the second component of the vector (denoted by φ_j^2) to be different from zero (and equal to unity), we arrive accordingly at the second row in these matrices, and finally at the third row by means of the vector φ_j^3. It is obvious that the calculations must be carried out successively for all points p_j.

The fundamental difficulties arise only in determining the matrices α_{jj} (with the same indices) because of the impossibility of putting the operator T_n under the integral sign. All other matrices can be constructed by using a quadrature formula. True, it should be remembered that if the indices j and k are such that the distance between the points p_j and p_k is comparable with the diameter of the region S_j, then, to improve the accuracy, we must make a second (finer) subdivision of the region S_j so as to apply the same quadrature formulas to smaller elements with the density held constant, of course, within the regions of the initial division. The size of these small elements is established in the course of calculations by attaining the required accuracy in the values of the coefficients being determined.

Let us now turn to the consideration of the central feature of the computational scheme, namely the construction of the matrices α_{jj}, and discuss the procedures for determining these coefficients proposed by different authors.

One of the recommendations (see P. I. Perlin, V. N. Samarov [1, 2]) is to erect a perpendicular to the surface at each pivotal point p_j and to lay off several points along it near the cut, which are denoted by p_j^l (the superscript l indicates the distance from the point p_j^l to the point p_j).

It is not difficult to construct, at each of the auxiliary points p_j^l, a matrix α_{jj}^l due to a potential extended over S_j and having a unit density (for each component in succession). Naturally, if the value of l is small, the application of the simplest quadrature formulas

involves a significant error. To achieve the required accuracy, it is necessary to use the method described above and subdivide the region S_j into small elements. After a set of such matrices is constructed (with warranted accuracy), it is suggested that the required matrix α_{jj} should be determined, say, by polynomial extrapolation (over the lengths l).

Naturally, if the matrices α_{jj} are to be determined within the required accuracy, the points p_j^l must be chosen sufficiently close to the point p_j and their number and position must be such as to ensure the validity of extrapolation.*

The authors give the results of calculations for a model example. A plane square of unit side is taken, and a vector function is prescribed over it whose two tangential components are zero and the normal component is unity. This function is regarded as the density of a double layer potential. Next the stresses generated by this potential are determined at points located along a normal through the centre of the square. For the chosen points specified by the distance l from the plane of the square, the calculations are performed for different, progressively finer auxiliary subdivisions (into n^2 equal parts) of the basic square. The results of the corresponding calcula-

Table 6

Normal Stress Component as a Function of the Distance from the Surface for Different Auxiliary Subdivisions

n \ l	0.40	0.20	0.15	0.10	0.05
30	1.78139	1.66073	1.63240	1.60990	1.51901
90	1.78159	1.66085	1.63255	1.61014	1.57513
120	1.78159	1.66093	1.63260	1.61023	1.59691
180	—	—	—	—	1.59615

tions are given in Table 6. It follows from these data that stable stress values can be obtained by using an auxiliary (sufficiently fine) subdivision, however small the distance of the point from the surface.

Further the process of extrapolation itself is studied by choosing different combinations and a different number of points (l_i). The values of the normal stress component thus obtained are given in Table 7.

* The points p_j^l are chosen on normals only for ease of calculation.

Table 7

Limiting Stress Values as a Function of the Number and Position of Auxiliary Points for Different Auxiliary Subdivisions

n \ l_j	0.4, 0.3, 0.2	0.4, 0.3, 0.2, 0.1	0.4, 0.3, 0.2, 0.1, 0.05
30	1.55197	1.59238	—
60	1.55195	1.59315	1.54165
120	1.55200	1.59388	1.59138
180	—	—	1.59205

By analysing these data it may be concluded that the proposed algorithm is stable if, of course, the stress values used are those supported by calculations with a finer subdivision.

Of course, the corresponding calculation at each pivotal point (particularly if the regions S_j are different) is very laborious, but it might be well to point out that in practice the choice of auxiliary points should be made once and for all from analysis of model cases (such as the square element considered above).

Thus, at the final stage the problem reduces to the solution of a system of algebraic equations:

$$\sum_{k=1}^{N} \alpha_{jk}\varphi_k = f(p_j) \quad (j=1,\ 2,\ \ldots,\ N). \tag{35.4}$$

As established in some special cases, system (35.4) is a fairly well-conditioned system, which can apparently be solved by the method of successive approximations.

In the work of B. M. Zinov'ev [1] a different procedure is suggested for constructing system (35.4). The author considers a plane square in space and prescribes a uniformly distributed load over it. By integrating the Kelvin-Somigliana solution, he obtains an expression in explicit closed form for the stresses in the whole space, and in particular for the limiting stress values from the two sides of the square at its centre. By summing the solutions obtained in the case of two closely spaced squares, and applying a limiting process as the distance between the squares tends to zero and the forces tend to infinity (so that the product remains constant), the author virtually finds an exact value of the operator $\boldsymbol{T}_n$ when the density is a unit vector and the surface S_j is a plane square. As an example, the initial cut is taken in the form of a plane rectangle with a side ratio of 2 : 1. Figure 18*a* gives values of the density (i.e., of the displacements) at

the points of the cut*, and Fig. 18b values of the normal stresses at points of the elastic body situated on the prolongation of the cut.

H. Nisitani and Y. Murekami [1] also consider the case of a cut of plane shape, and the authors' reasoning is as follows. Suppose that there is a plane cut in the form of an ellipse. The solution of the problem for a space with an elliptical cut under hydrostatic pressure

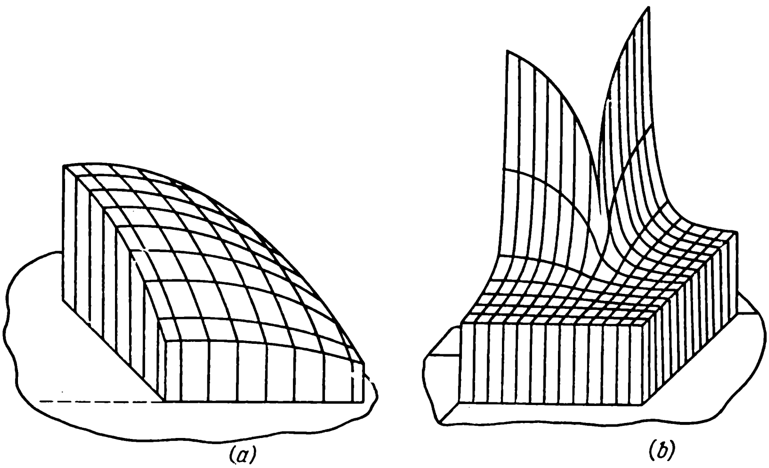

Fig. 18. Space with a cut in the form of a rectangle
(a) displacements on the surface of the cut; (b) normal stresses in the plane of the cut

is known (see A. I. Lur'e [1]). Let us solve the problem numerically for a given arbitrary normal pressure (since there is a plane of symmetry it follows that the tangential components of the vectors φ_j are zero). The surface of the cut is divided in some way and then system (35.4) is constructed. To find the diagonal matrices α_{jj}, the authors proceed as follows. The problem is considered in an approximate manner for the case where the solution is known in explicit form, and these diagonal elements are found on the basis that all off-diagonal terms of the matrix are calculated without difficulty and the values of the numbers φ_h are taken from the exact solution.

It is natural to expect that the proposed algorithms lead to a readily practicable computational scheme when the densities may be assumed constant within sufficiently large elementary regions. Such an assumption is considered logical only in the interior regions if it is supposed that the curvature of the surface and the load vary rath-

* By the formulatin of the problem, the density must vanish at the points of the contour L.

er slowly. In a narrow strip adjacent to the contour L the solution varies considerably.

We use the well-known result in elasticity (see Sec. 17) to the effect that the state of stress in the neighbourhood of the edge of a wedge-shaped region is satisfactorily described by asymptotic solutions derived from the corresponding plane and torsion problems. Let an annular strip S_δ be isolated from the surface S in which the solution is represented by the above solutions introduced in a local co-ordinate system along the normal and tangent to the contour of the cut with a certain factor of proportionality varying along the length of the contour (see P. I. Perlin, V. N. Samarov [1, 2]). This factor is defined so that at the nearest (along the normal to the contour pivotal point from the remainder of the region $S^* = S - S_\delta$ the jump in the displacements determined from the analytic solution coincides with an arbitrarily assumed value $0.5\varphi_j$. All calculations are then performed only at the pivotal points situated in the zone S^*, but the integration is extended over the entire region S. After system (35.4) (or, more precisely, its modification) is solved, the displacements in the strip are determined analytically.

As an example, consider an axially symmetric problem for a circular cut of radius R. The surface is divided into small regions by a system of equidistant rays and concentric circles, and the pivotal points are taken at the centres (using a polar co-ordinate system). The solution in the ring S_δ is chosen in the form $\varphi(\rho) = \varphi_k \sqrt{(R-\rho)/(R-\rho_k)}$, where k is the subscript of the outer pivotal point situated in the region S^*, and ρ and ρ_k are the distances of the current points p and p_k from the centre (because of axial symmetry all con-

Table 8

Values of the Normal Displacement Component on the Surface of a Circular Cut

ρ	1	1/30	5/30	10/30	15/30
φ_{cal}	1.10864	1.10203	1.09527	1.05008	0.95484
φ_{exact}	1.09645	1.09430	1.08020	1.06370	0.96118

ρ	20/30	25/30	28/30	29/30
φ_{cal}	0.82501	0.58735	0.35894	0.20987
φ_{exact}	0.82030	0.59326	0.36148	0.21051

structions are performed in one sector). As the calculation data show* (see Table 8), the value of the intensity factor increases in accuracy from 5 to 1.5 per cent when using asymptotic solutions with the same discretization of the surface.

We now turn to the study of problems involving a finite elastic body. Here two cases may be distinguished. In the first the surface of a cut is assumed to lie inside the elastic body, and in the second it approaches the outer surface S_1 over a certain portion. It is obvious that although the solution demands considerable labour in these cases, the significance of the fundamental difficulties involved should not be exaggerated. In contrast to the problem considered above for the whole space, the representation of the displacement is now of the form

$$U(p)=\int_S \Gamma_2^{\mathrm{I}}(p,\ q)\,\varphi(q)\,dS_q+H(p), \tag{35.5}$$

where the term $H(p)$ is a simple layer or double layer potential on the surface S_1 (depending on the nature of the boundary condition on it). Supposing in the following discussion that stresses $f_1(q)$ are prescribed, we represent the boundary condition on the surface S_1 in terms of the function $\varphi(q)$ assumed to be given:

$$\varphi_1(q)+\int_{S_1}\Gamma_1(q,\ q')\,\varphi_1(q')\,dS_{q'}=$$
$$=f_1(q)-\int_S T_n\Gamma_2^{\mathrm{I}}(q,\ q')\,\varphi(q')\,dS_{q'}\quad (q\in S_1). \tag{35.6}$$

If N small regions S_j are introduced on the surface S, it is necessary to solve $3N$ boundary value problems (i.e., equations of the form of (29.3)) for the uncut region setting only one component of one of the vectors φ_j different from zero in succession. Further the reasoning used in the first part of the present section should be repeated adding the regular terms $H(p)$ in constructing the matrix.

We may also propose a method based on the approach of P. I. Perlin [2, 3]. For simplicity, the boundary conditions are assumed to be zero on the surface S_1, where an auxiliary function $\varphi_2(q)$ is introduced equal to half the displacement on this surface. We form a double layer potential $W(p,\ \varphi_2)$ and consider a displacement $U_1(p)=U(p)-W(p,\ \varphi_2)$, which satisfies the equations of elastic equilibrium in the entire space, excluding the surface S. We thus arrive at the problem considered above. The corresponding functional equation has additional terms depending on the particular form of representation of the function $\varphi_2(q)$ on the surface S_1. In general, a piecewise constant representation of this function can be used.

* Hydrostatic load.

Note that in the case when the elastic body occupies a half-space the problem can be treated in a special way. In constructing the functional equation, it is well to proceed not from the Kelvin-Somigliana solution, but from Mindlin's solution for a concentrated force in a half-space (see R. D. Mindlin [1]). The boundary conditions on the outer surface (which must be set equal to zero) are satisfied exactly. This has been used to obtain a solution of the problem for

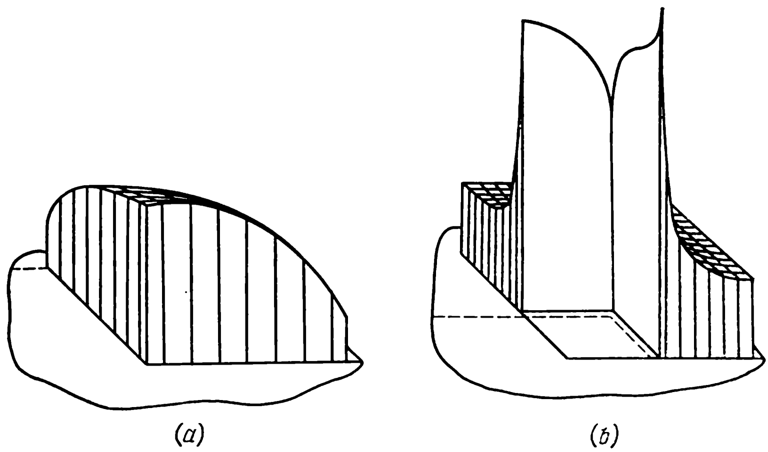

Fig. 19. Half-space with an external cut in the form of a rectangle
(a) displacements on the surface of the cut; (b) normal stresses in the plane of the cut.

a half-space with a cut in the form of a rectangle (see B. M. Zinov'ev [1]). Figure 19 gives values at points of the elastic body situated on the prolongation of the cut.

Let us now consider the problem when the cut approaches the outer surface. The part of the contour L situated on the surface S_1 is denoted by L_1. By introducing an auxiliary potential on the surface S, we pass to a problem for the uncut region and, solving it in some way or other (assuming, for example, the jump in the displacements on S to be arbitrarily given), we arrive at an equation for the jump at the final stage. The first serious difficulty arises from the fact that we have to solve the auxiliary problem for the uncut body with a discontinuous boundary condition along the line L_1. Moreover, as noted in Sec. 17, there is no information of the asymptotic behaviour of the solution in the neighbourhood of the ends of this line.

A large number of works relate to the case when a cut differs only slightly (in a sense) from a circle. The methods of constructing and solving the corresponding integral equations will be discussed in greater detail in Sec. 38 with reference to contact problems.

Let us consider an approximate method of solving problems for bodies with cuts (V. Z. Parton, P. I. Perlin [2]). The cut is replaced by a sufficiently thin cavity with the same boundary conditions prescribed on its surface. Suppose that the solution of this problem has been obtained. Let us try to derive information from it regarding the solution of the original problem, and in particular to determine the stress intensity factor.* It will be recalled that the asymptotic behaviour of the stresses on the prolongation of the cut in a local co-ordinate system in a plane perpendicular to the edge of the cut is as follows:

$$\sigma_z = \frac{K}{\sqrt{\rho}}. \tag{35.7}$$

It is suggested that the constant K (depending on the position of the point of the edge under consideration) should be determined from the condition that the stresses obtained from the solution for a cavity are fairly well approximated by formula (35.7). If this approach is followed, it is necessary to allow arbitrariness in the position of the origin of the local co-ordinate system used in the asymptotic expansion (35.7). This arbitrariness seems natural since in passing to a cavity one must assume small changes in size of the cut in plan. The approximation used is, in fact, of the form

$$\sigma_z = \frac{K}{\sqrt{\rho - \varepsilon}}, \tag{35.8}$$

where ε defines the change mentioned above.

If on performing several calculations (with decreasing thickness of the cavity) the stability of values of the constant K is observed (for small values of ε), we may speak of a sufficiently valid solution of the problem for a body with a cut.

In the work cited above the solution of the auxiliary problem was obtained by using the singular integral equations (29.3). It is obvious that the convergence of the algorithm becomes worse with decreasing thickness of the cavity, but, as calculations show, a sufficient degree of accuracy can be achieved for the stress intensity factor K.

36. Piecewise Homogeneous Bodies

We first assume that there is a cavity filled with an elastic medium having Lamé's constants λ_0 and μ_0 in an infinite elastic body with different constants, λ_1 and μ_1. The surface of contact between the media is denoted by S, and Poisson's ratio of these media is assumed to be the same; we have the following equality: $\lambda_0/\lambda_1 = \mu_0/\mu_1 = k$.

* In the plane case the required result can be obtained by using Stern's approach (see Sec. 25).

Let the displacement in the region D^- be denoted by $\boldsymbol{U}_1(p)$, and in the region D^+ by $\boldsymbol{U}_0(p)$.

Suppose that the following conditions are fulfilled on the surface S:

$$\boldsymbol{U}_1(q) - \boldsymbol{U}_0(q) = \boldsymbol{F}_1(q), \tag{36.1}$$

$$\boldsymbol{T}_{1n}\boldsymbol{U}_1(q) - \boldsymbol{T}_{0n}\boldsymbol{U}_0(q) = \boldsymbol{F}_2(q), \tag{36.2}$$

where the functions $\boldsymbol{F}_1(q)$ and $\boldsymbol{F}_2(q)$ belong to the class H-L. The additional subscript in the stress operator corresponds to the subscript on Lamé's coefficients. In the terminology of V. D. Kupradze, T. G. Gegelia et al. [1], this problem is called *the principal contact problem.*

In accordance with the work of P. I. Perlin [7] we modify the formulation of the problem by making use of the fact that Poisson's ratio for the elastic media is the same. Let the elastic medium in either of the regions, say in D^+, be replaced by a body with the same constants as in the region D^- keeping the displacements unchanged. Such a replacement is permissible since homogeneous Lamé's equations contain only Poisson's ratio. Equation (36.2) then becomes

$$\boldsymbol{T}_{1n}\boldsymbol{U}_1(q) - k\boldsymbol{T}_{1n}\boldsymbol{U}_0(q) = \boldsymbol{F}_2(q), \tag{36.3}$$

while Eq. (36.1) remains unaltered.

To simplify the problem, we first form a double layer potential $\boldsymbol{W}(p, {}^1/_2\, \boldsymbol{F}_1)$ and consider the displacements $\widetilde{\boldsymbol{U}}_1 = \boldsymbol{U}_1(p) - \boldsymbol{W}(p, {}^1/_2\, \boldsymbol{F}_1)$ and $\widetilde{\boldsymbol{U}}_0(p) = \boldsymbol{U}_0(p) - \boldsymbol{W}(p, {}^1/_2\, \boldsymbol{F}_1)$. It is obvious that these displacements satisfy Eq. (36.1) identically, and Eq. (36.3) takes the form

$$\boldsymbol{T}_{1n}\widetilde{\boldsymbol{U}}_1(q) - k\boldsymbol{T}_{2n}\widetilde{\boldsymbol{U}}_0(q) = \boldsymbol{F}_2(q) + k\boldsymbol{T}^+_{1n}\boldsymbol{W} - \boldsymbol{T}^-_{1n}\boldsymbol{W} = \boldsymbol{F}_3(q). \tag{36.4}$$

Thus, the new displacements $\widetilde{\boldsymbol{U}}_1(p)$ and $\widetilde{\boldsymbol{U}}_0(p)$ are continuous functions in the entire space, and the limiting values of the stress operator of these functions are related by Eq. (36.4).

We introduce an auxiliary function $\boldsymbol{\varphi}(q)$ on the surface S defined as follows:

$$\boldsymbol{T}_{1n}\widetilde{\boldsymbol{U}}_1(q) - \boldsymbol{T}_{1n}\widetilde{\boldsymbol{U}}_0(q) = 2\boldsymbol{\varphi}(q). \tag{36.5}$$

If the function $\boldsymbol{\varphi}(q)$ were known, the solution of the problem would be represented in the form of a simple layer potential, namely

$$\boldsymbol{V}(p, \boldsymbol{\varphi}) = \int_S \boldsymbol{\Gamma}(p, q)\,\boldsymbol{\varphi}(q)\,dS_q. \tag{36.6}$$

This function is arbitrarily assumed to be given. By determining the limiting values of the stress operator from the two sides of the surface S, and substituting them in the true relation (36.4), we arrive at

an integral equation for the function $\varphi(p)$:

$$\varphi(q)-\frac{1-k}{1+k}\int_S \mathbf{\Gamma}_1(q,\ q')\,\varphi(q')\,dS_{q'}=\frac{1}{1+k}\,\boldsymbol{F}_3(q). \qquad (36.7)$$

Equations of this class have been investigated in Sec. 29. Since $\nu_0 = (1-k)/(1+k) < 1$, Eq. (36.3) is always solvable, and its solution can be obtained by using the method of successive approximations (see V. D. Kupradze [3]). The method can be implemented by means of the regular representations introduced in Sec. 31. The fact that the parameter $\nu_0 < 1$ ensures rapid and stable convergence of the algorithm.

Suppose that the elastic body with the constants λ_1 and μ_1 does not extend to infinity, but is bounded by a certain surface S_1 (it may equally be assumed that the elastic body with the parameters λ_0 and μ_0 is bounded from the inside by a certain surface S_0). By repeating the reasoning used above (and taking into account the results of Sec. 29), we arrive at a system of singular integral equations:

$$\varphi(q)-\nu_0\int_S \mathbf{\Gamma}_1(q,\ q')\,\varphi(q')\,dS_{q'}+$$

$$+\frac{1}{(1+k)}\int_{S_1} \mathbf{\Gamma}_1(q,\ q')\,\varphi_1(q')\,dS_{q'}=\frac{1}{(1+k)}\,\boldsymbol{F}_3(q) \quad (q\in S), \qquad (36.8)$$

$$\varphi_1(q)+\int_S \mathbf{\Gamma}_1(q,\ q')\,\varphi(q')\,dS_{q'}+$$

$$+\int_{S_1} \mathbf{\Gamma}_1(q,\ q')\,\varphi_1(q')\,dS_{q'}=\boldsymbol{F}_4(q) \quad (q\in S_1). \qquad (36.9)$$

It has been assumed above that stresses $\boldsymbol{F}_4(q)$ are prescribed over the surface S_1, and in consequence the representation of the solution is chosen as a simple layer potential. Equations (36.8) and (36.9) resemble those obtained in Sec. 31 for a body bounded by two surfaces.

We may approach the integral equation (36.7) on the basis of the general functional equations for a piecewise homogeneous medium, which are also suitable in the case of different Poisson's ratios (see V. D. Kupradze [3]). Proceeding from these equations, the author has proved the solvability of the original physical problems.

We now turn to the consideration of the general case (with different Poisson's ratios) and assume that there is a region D bounded by smooth surfaces S_n ($n = 0, 1, 2, \ldots, m$) one of which, S_0, encloses all the others. The region D is filled with an elastic medium having Lamé's constants λ_0 and μ_0, and the regions D_n^+ contained within the surfaces S_n are filled with media having Lamé's constants λ_n and μ_n ($n = 1, 2, \ldots, m$). The conditions on the surfaces of contact

between the media are of the type of (29.1) and (29.2), and the conditions on the surface S_0 are those for the first or second fundamental problem. Let the stress vectors on the surfaces S_n be denoted by $\boldsymbol{t}_n^{\pm}(q)$, and the displacement vectors by $\boldsymbol{U}_n^{\pm}(q)$. Following F. J. Rizzo and D. J. Shippy [1], we apply, to each of the regions D_n, relation (31.5) derived from Betti's formula (in its original form and in the modified form for the case of several surfaces). The result is

$$\boldsymbol{U}_n^{+}(q)-\int_{S_n}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_n^{+}(q')\,dS_{q'}=\int_{S_n}\boldsymbol{\Gamma}(q,\ q')\,\boldsymbol{t}_n^{+}(q')\,dS_{q'}\qquad(n\leqslant m),$$

$$\begin{aligned}\boldsymbol{U}_n^{-}(q)&+\int_{S_n}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_n^{-}(q')\,dS_{q'}-\\&-\sum_{j=1}^{m}{}'\int_{S_j}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_j^{-}(q')\,dS_{q'}-\int_{S_0}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_0^{+}(q')\,dS_{q'}=\\&=\sum_{j=1}^{m}\int_{S_j}\boldsymbol{\Gamma}(q,\ q')\,\boldsymbol{t}_j^{-}(q')\,dS_{q'}+\\&+\int_{S_0}\boldsymbol{\Gamma}(q,\ q')\,\boldsymbol{t}_0^{+}(q')\,dS_{q'}\qquad(n=1,\ 2,\ \ldots,\ m),\end{aligned}\tag{36.10}$$

$$\begin{aligned}\boldsymbol{U}_0^{+}(q)&+\sum_{j=1}^{m}\int_{S_j}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_j^{-}(q')\,dS_q-\\&-\int_{S_0}\boldsymbol{\Gamma}_2^{\mathrm{I}}(q,\ q')\,\boldsymbol{U}_0^{+}(q)=\int_{S_0}\boldsymbol{\Gamma}(q,\ q')\,\boldsymbol{t}_0^{+}(q')\,dS_{q'}.\end{aligned}$$

A prime on the summation denotes the omission of the nth term.

Let the above system be supplemented by two equations of the form of (36.1) and (36.2). We finally obtain a system of order $4m+1$, which, because of the simplicity of relations (36.1) and (36.2), is simplified at once and reduces to a system of order $2m+1$. It is obvious that the order of unknowns can be lowered in various ways, giving regular equations of the first kind or singular equations of the second kind.

In the same work the case of the plane problem is considered in detail using representation (29.19). A scheme of numerical implementation is described as a development of the work of F. J. Rizzo [1]. Below are given some results of this work. The authors have considered the problem of determining the state of stress in a plane with an elliptical hole into which an elliptical disk of the same material is inserted with negative allowance (Fig. 20). The figure shows the division of the contour into elementary arcs and the choice of centre points for the twelve sections shown.

Table 9 gives values of the relative error for the displacement component (Δ_1) and the stress component (Δ_2) normal to the contour at the point 1 and, respectively, at the points 4, 7, 13 when the number of divisions is $n = 12, 24, 48$. The error is determined by com-

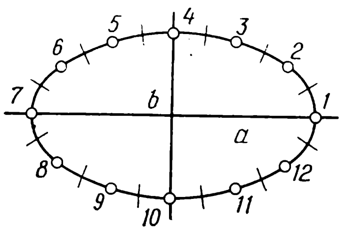

Fig. 20. Infinite plate with an elliptical inclusion

parison with the exact solution (see D. I. Sherman [5]) and expressed as fractions of the maximum displacement and the maximum stress. In calculations, Poisson's ratio is taken to be $\nu = 1/3$. The negative allowance is specified by an increase of the semiaxes in proportion to their size. The ratio of the semiaxes is $a/b = 2$.

Table 9

Error in Displacements and Stresses for a Plane with an Elliptical Inclusion

Points	Δ_1			Δ_2		
	12	24	48	12	24	48
1	0.0220	0.0115	0.0055	0.1560	0.0426	0.0113
4	0.0160	—	—	0.0266	—	—
7	—	0.0095	—	—	0.0133	—
13	—	—	0.0005	—	—	0.0066

The authors have also considered the problem of an elliptical insert in a matrix bounded externally by a circumference R (Fig. 21). Both the insert and the matrix are, as before, of the same material. Table 10 gives values of the components of the displacements and stresses at the interface and on the outer boundary. These values are given in non-dimensional form as fractions of the maximum displacements and stresses corresponding to the case when the outer contour is absent ($R/a = \infty$). All other parameters correspond to the preceding problem.

Besides, the authors have considered the problem of an infinite plane with a square hole and an insert fitted into it with negative allowance when the insert is of the same material or of a material

with an elastic modulus three times that of the matrix (Fig. 22). The contour is divided into arcs of equal length so that the first centre point is nearest the axis of symmetry (but lies off the axis). Moreover, the corner point must not be a centre point. Table 11 gives values of the normal and tangential components of the contact stresses

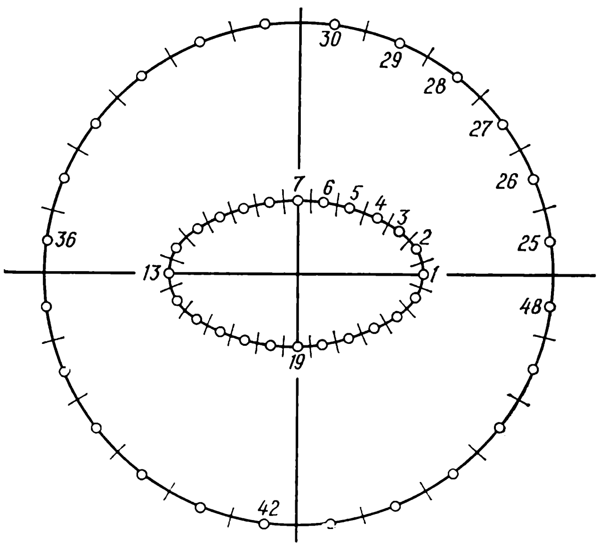

Fig. 21. Circular plate with an elliptical inclusion

and also of the tangential component in both the inclusion (σ_θ) and the matrix (σ'_θ) when the elastic moduli are the same. All quantities are given in non-dimensional form (divided by the normal stress

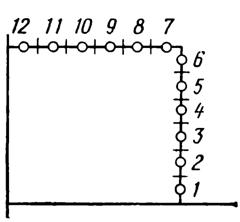

Fig. 22. Plate with a square inclusion

component at the point 1 determined from the analytic solution). The negative allowances are taken to be the same in both directions. Numbers in parentheses indicate a relative error determined from the exact solution (as fractions of the same normal stress component). Poisson's ratio is $\nu = 0.25$.

Table 10

Displacements and Stresses for a Circle with an Elliptical Inclusion

Points	u_1	u_2	σ_n	τ_n
$R/a=2$				
1	0.623	0.000	0.9115	0.0000
2	0.577	0.421	0.7396	0.3414
3	0.481	0.718	0.4911	0.4650
4	0.368	0.920	0.3239	0.5065
5	0.249	0.048	0.1983	0.5243
6	0.125	1.121	0.0968	0.5338
7	0.000	1.146	0.0000	0.5341
25	0.217	0.044	0.0000	0.0000
26	0.255	0.154	0.0000	0.0000
27	0.281	0.288	0.0000	0.0000
28	0.256	0.488	0.0000	0.0000
29	0.178	0.585	0.0000	0.0000
30	0.064	0.632	0.0000	0.0000

Points	u_1	u_2	σ_n	τ_n
$R/a=5$				
1	0.627	0.000	0.9581	0.0000
7	0.000	0.669	0.0000	0.6369
$R/a=\infty$				
1	0.634	0.000	0.9660	0.0000
7	0.000	1.014	0.0000	0.6574
Exact solution ($R/a=\infty$)				
1	0.615	0.000	1.0000	0.0000
7	0.000	1.000	0.0000	0,6677

Table 11

Stresses in a Plate with a Square Inclusion (Material of the Plate and Inclusion Is the Same)

Points	σ_n	τ_n	σ_0	σ'_θ
1	0.987 (0.012)	0.061 (0.000)	1.831 (0.005)	1.005 (0.005)
2	0.992 (0.011)	0.185 (0.000)	1.827 (0.003)	1.008 (0.005)
3	0.998 (0.013)	0.327 (0.000)	1.819 (0.0005)	1.016 (0.005)
4	1.009 (0.015)	0.501 (0.002)	1.808 (0.003)	1.027 (0.005)
5	1.026 (0.011)	0.726 (0.027)	1.805 (0.008)	1.028 (0.008)
6	0.988 (0.065)	0.525 (0.256)	1.810 (0.028)	1.024 (0.030)

Table 12 gives the results of similar calculations when the elastic modulus of the inclusion is three times that of the matrix. For convenience in comparing the results, all quantities are given in nondimensional form (using the same quantity as in the preceding table).

The case where there is one interface in a finite region is specially considered in the work of M. O. Basheleishvili and T. G. Gegelia [1], where it is assumed that stresses are prescribed on the outer

Table 12

Stresses in a Plate with a Square Inclusion (Material of the Plate and Inclusion Is Different)

Points	σ_n	τ_n	σ_θ	σ'_θ	Points	σ_n	τ_n	σ_θ	σ'_θ
1	1.142	0.092	2.531	1.738	4	1.225	0.765	2.618	1.691
2	1.153	0.285	2.541	1.732	5	1.307	1.100	2.765	1.623
3	1.177	0.499	2.565	1.719	6	1.659	2.405	2.776	1.541

surface. By a suitable choice of representations for displacements in each region, a system of singular integral equations for four unknown functions is obtained and the applicability of the Fredholm alternatives to it is proved.

In cases where the interface extends to the boundary the problems are essentially more complicated. The foregoing methods can apparently be used (with certain assumptions) to solve these problems, but, of course, with due analysis of singularities for the state of stress and strain in the neighbourhood of the lines of intersection of the outer surface and the surface of the cut.

Consider a special but very important case for applications, namely the case where all elastic constants are the same. For simplicity, we assume that there is only one inclusion. We proceed from the conditions

$$\left.\begin{aligned} U_0(q) - U_1(q) &= F_1(q) \\ T_nU_0(q) - T_nU_1(q) &= 0 \end{aligned}\right\} \quad (q \in S_1). \tag{36.11}$$

It may now be assumed that the surface S_1 is not closed and approaches the surface S. Problems of this kind arise, in particular, in engineering applications when one body (of somewhat larger size) is inserted into a cavity of another.

We form a simple layer potential $W(p, {}^1/_2F_1)$. Obviously, it follows from (28.10) that the displacements

$$\begin{aligned} U(p) &= U_0(p) - W(p, {}^1/_2F_1) \quad (p \in D) \\ U(p) &= U_1(p) - W(p, {}^1/_2F_1) \quad (p \in D_1^+) \end{aligned} \tag{36.12}$$

satisfy Lamé's equations in the entire region. The boundary conditions for the displacement $U(p)$ are determined both by those (specified in the formulation of the problem and by the potential $W(p, {}^1/_2F_1)$. After solving this problem in some way or other, it is necessary to

return to the displacements* $U_0(p)$ and $U_1(p)$ according to (36.12). Some difficulty in implementing the foregoing method occurs when determining the stresses on the contact surface since in this case we have to evaluate the stress operator of a double layer potential. We

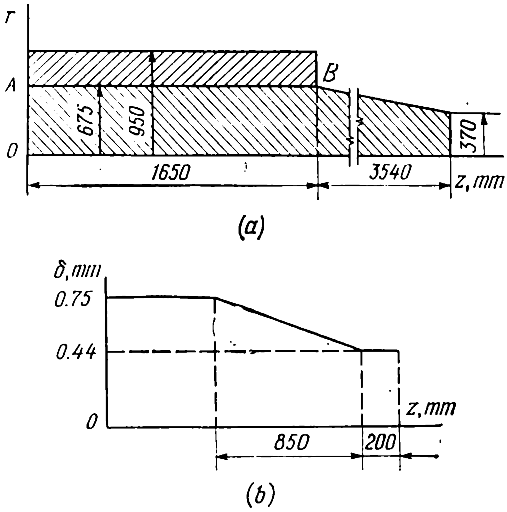

Fig. 23. Section of a quarter of a compound roll (*a*) and profiling of the spindle (*b*)

may determine displacements using the regular representations (31.17) and transform to stresses. We may also use the extrapolation procedure discussed in detail in Sec. 35.

V. P. Polukhin, V. G. Kostylev and N. F. Andrianov [1] have obtained the solution of an axially symmetric problem arising in

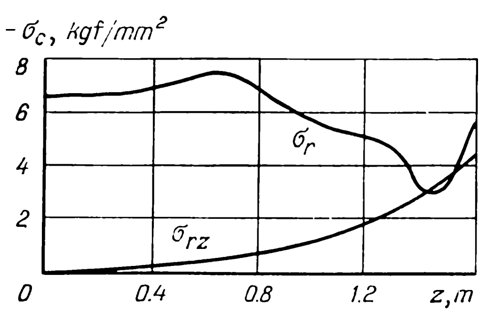

Fig. 24. Contact stresses

the analysis of the state of stress in a sleeved roll of a rolling mill, which occurs when a hollow cylinder (sleeve) is shrinked on a solid cylinder of variable diameter (spindle). The jump in the normal

* It might be well to point out the resemblance between this procedure and the approach used in the plane problem (see Sec. 22).

displacement component is determined by the original dimensions of the bodies. If there is full cohesion on cooling, it is not difficult to calculate the jump in the tangential displacement component from the value of the linear expansion coefficient: the amount of the jump in uniform heating is a linear function of the axial coordinate. However, as experiments show, slipping may occur on the contact surface. It is therefore assumed in the cited work that the jump is a linear function (as in the case of cohesion), but of half its amount.

Figure 23*a* and *b* shows the section of compound roll and the profiling of its spindle (i.e., the amount of the jump in the normal displacement component). Figure 24 gives contact stress diagrams, and Fig. 25 stress diagrams for the spindle and sleeve at four sections: at the central section and at sections distant 650, 1150 and 1450 mm from it.

Note that if the surface of contact between the bodies is cylindrical and there is a jump (of constant amount) only in the normal component of the displacement, we can pass to the problem for a solid body directly by superimposing the solutions of Lamé's problem.

37. Solution of Problems of the Theory of Elasticity for Bodies Bounded by Piecewise Smooth Surfaces

So far it has been assumed that the surfaces bounding elastic bodies are Lyapunov surfaces. This fact has been used, in particular, in the derivation of integral equations.

Let us now discuss in greater detail some general questions of the formulation of boundary value problems in the case of irregular boundaries. Suppose that the surface contains edges, conic points, vertices of polyhedral angles. The complete surface can then be represented as a set of sufficiently smooth unclosed surfaces, which intersect forming edges and vertices. In the formulation of the first boundary value problem the initial data may be prescribed over the entire boundary. In the case of the second problem the boundary conditions must be prescribed only at the interior points of each of the smooth pieces of the surface (since the formulation of the problem involves the direction cosines of a normal). The irreguar points of the boundary can be considered only as the limiting points of a particular piece of the smooth surface. In this approach each point of an edge, for example, must be considered twice, namely as belonging to one surface and another (which influences the direction of a normal).

The questions of applying the potential method to the solution of harmonic problems for regions bounded by irregular boundaries have been discussed in the works of T. Carleman [1], J. Radon [1], Yu. D. Burago, V. G. Maz'ya and V. D. Sapozhnikova [1] and others. Similar questions in reference to equations among which are the equations of the theory of elasticity have been studied in the work of

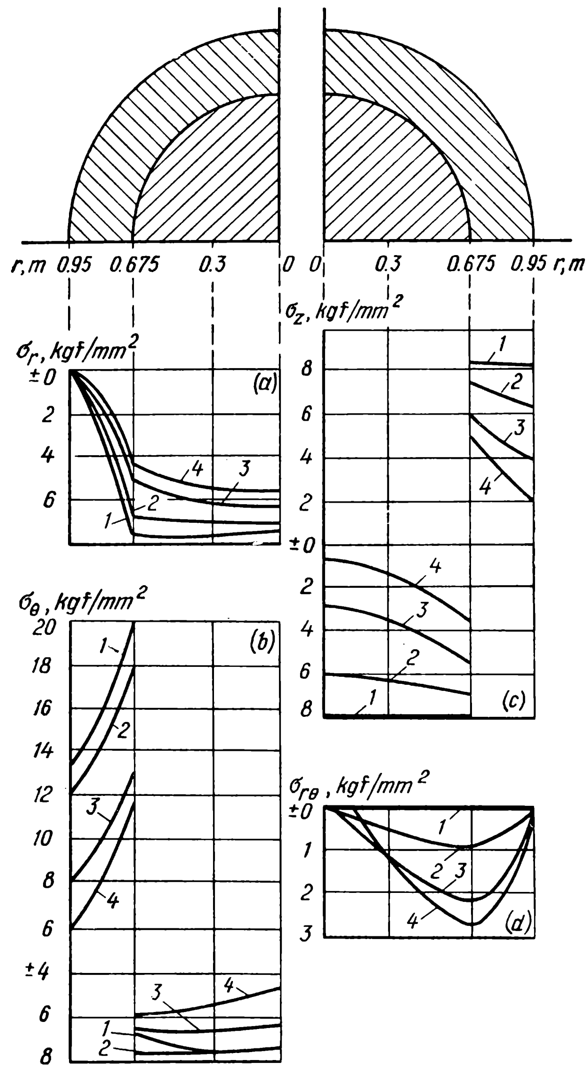

Fig. 25. Stress diagram for a compound roll

1—at the central section; 2, 3, 4—at sections distant 1.0, 0.5 and 0.2 m from the sleeve edge

G. I. Kresin and V. G. Maz'ya [1]. It should be noted, however, that no use has been made of the above approach for the actual solution of boundary value problems in elasticity.

On the other hand, the question of the behaviour of the solutions of boundary value problems (in terms of displacements and stresses) in the neighbourhood of irregular points of the boundary has been fairly well studied (see Sec. 17). It seems, therefore, justifiable to try to extend formally* the algorithms described in Sec. 33 to the case of these surfaces with a proper control based on the known information about the properties of the solution (known before the construction of the solution itself).

In the work of N. F. Andrianov and P. I. Perlin [1] the solution has been constructed by successive approximations using both pivotal and nodal points in their implementation. In this case the discretization of each of the smooth pieces of the surface must be made separately to avoid the appearance of elementary regions with irregular points inside. All pivotal points are therefore situated in the smooth portions of the boundary, and hence the evaluation of cubatures can be performed by the same formulas as in the case of smooth surfaces (including the use of regular representations). The difference is that the functions $\varphi_n(q)$ at nodal points situated at an edge (each of them is considered twice) are determined by extrapolation (and not by interpolation as in the case of nodal points situated in smooth portions) using the values at the nearest pivotal points both on the one surface and the other. Naturally, different values will result. Thus, the edge is a line of discontinuity for the unknown density of the simple layer potential.

If we proceed from a simpler computational scheme using the rectangular formula (only pivotal points are employed in the implementation), no question of extrapolation arises. In this case the discretization must be made as described above. The discontinuity in density across an edge occurs automatically here.

Of course, the presence of irregular ponts of the boundary gives rise to an abrupt increase in the density $\varphi(q)$ in their neighbourhood, and hence the construction of a convergent computational scheme may require a much finer discretization than in the case of smooth surfaces. Note that a fine discretization is also needed for smooth portions when the curvature changes considerably.

Calculations show that with an appropriate discretization the algorithms described in Sec. 33 become convergent, and the stability of values of the density is observed at points at a small distance from irregular points. This leads to stable stress values everywhere except in a small neighbourhood of these points.

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* Without rigorous mathematical proof.

The availability of information on the behaviour of the solution of a boundary value problem in the neighbourhood of irregular points must be utilized to study the asymptotic behaviour and, in general, to estimate the error of the approximate solution.

The foregoing considerations will be illustrated by an analysis of the solution of a model problem.

Let the surface be formed by revolving a square of unit side about its diagonal. There are a conic point and an edge on the surface. The solution of the (exterior and interior) problems is obtained for a unit hydrostatic pressure. The discretization of the contour (the problem is axially symmetric) is specified by the position of pivotal points (the nodal points are midway between them). Table 13 gives values of the distances of ten pivotal points nearest the vertex, and Table 14 the distances of ten pivotal points nearest the edge.

Table 13

Points	1	2	3	4	5	6	7	8	9	10
ρ	0.0001	0.0002	0.0003	0.0004	0.0005	0.0007	0.0010	0.0015	0.0020	0.0040

Table 14

Points	1	2	3	4	5	6	7	8	9	10
ρ	0.0001	0.0002	0.0003	0.0004	0.0005	0.0007	0.0010	0.0015	0.0025	0.0030

The values of the density obtained in the analysis are given in Tables 15 and 16. The data of Table 15 refer to the points near the vertex, and the data of Table 16 to those near the edge.

Table 15

Points	1	2	3	4	5	6	7	8	9	10
φ_r^+	1.614	1.269	1.116	1.026	0.957	0.872	0.798	0.727	0.659	0.582
φ_z^+	10.41	8.03	6.88	6.16	5.64	4.95	4.30	3.67	3.27	2.51
φ_r^-	5.28	4.44	3.96	3.78	3.61	3.39	3.17	2.94	2.80	2.49
φ_z^-	1.59	1.43	1.29	1.21	1.18	1.11	1.05	0.988	0.989	0.823

Table 16

Points	1	2	3	4	5	6	7	8	9	10
φ_r^+	2.78	2.30	2.08	1.95	1.85	1.71	1.58	1.45	1.31	1.26
φ_z^+	0.53	0.58	0.55	0.53	0.51	0.50	0.47	0.45	0.44	0.43
φ_r^-	0.686	1.17	1.01	0.91	0.84	0.78	0.73	0.675	0.647	0.62
φ_z^-	31.94	22.67	18.67	16.33	14.75	12.67	10.81	9.05	7.24	6.69

The final results of the calculation, viz. the stresses, are presented in Tables 17 and 18. Table 17 gives stress values at points on the axis of revolution (as a function of the distance from the vertex). Also given are the values of the function $\rho^{-0.2}$ obtained from

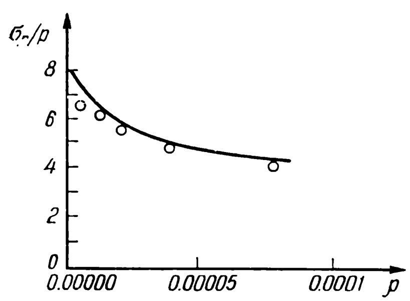

Fig. 26. Stress σ_r on the axis of revolution

Eq. (17.16) according to Fig. 8 for $\nu = 0.3$ (then $\gamma = 0.8$). In the last lines of the table are the ratios of the calculated components of the stress tensor to the function $\rho^{-0.2}$. The results of the calculation for the component σ_r are given in Fig. 26. Table 18 is arranged in a similar way. Here are given the stress components, the function $\rho^{-0.455}$ obtained according to representation (17.7) from Eq. (17.10) for $\gamma = 0.545$, and the ratios of all stresses to this function.

The results of the calculation for the component σ_z are given in Fig. 27.

It follows from the data given above that each of the ratios obtained has a maximum, and this suggests a procedure for establishing the asymptotic properties (the coefficients of the eigenfunctions). We first consider Table 17. Let C_r (0.59) and C_z (0.50) be the values

Table 17

ρ	0.00001	0.00002	0.00004	0.00006	0.00010
$\sigma_z = \sigma_\theta$	5.86	5.07	4.22	3.79	3.33
σ_z	4.96	4.29	3.28	2.72	2.23
$\rho^{-0.2}$	10	8.705	7.578	6.988	6.310
$\sigma_r\rho^{0.2}$	0.586	0.582	0.557	0.542	0.527
$\sigma_z\rho^{0.2}$	0.496	0.493	0.428	0.389	0.353

of the corresponding maxima. The points at which the maximum is reached are considered to be the points determining the transition from the solution obtained in an approximate manner (in the present case on the basis of the integral equation) to the asymptotic

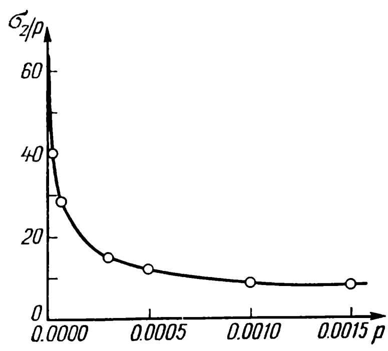

Fig. 27. Stress σ_z in the middle plane

solution; the factors multiplying the eigenfunctions are the ratios C_r and C_z established above. This choice of transition points and coefficients ensures a smooth transition from one solution to the other.

Consequently, it may be stated on the basis of the calculations that

$$\sigma_\theta = \sigma_r = 0.59\rho^{-0.2}, \quad \sigma_z = 0.5\rho^{-0.2}. \tag{37.1}$$

By using Eqs. (17.17), it is not difficult to obtain the asymptotic solution in an arbitrary direction.

Table 18

ρ	0.00001	0.00002	0.00004	0.00008	0.00010	0.00015
σ_r	24.78	24.02	20.83	17.88	14.10	11.41
σ_θ	21.67	18.93	15.09	12.71	10.11	8.38
σ_z	46.46	38.46	28.92	23.92	18.97	15.77
$\rho^{-0.455}$	188.4	137.4	100.2	83.35	66.07	54.94
$\sigma_r\rho^{0.455}$	0.131	0.178	0.208	0.214	0.2134	0.208
$\sigma_\theta\rho^{0.455}$	0.115	0.137	0.151	0.152	0.1530	0.1525
$\sigma_z\rho^{0.455}$	0.247	0.280	0.289	0.287	0.2871	0.2870

We now turn to the analysis of the solution in the neighbourhood of the edge (Table 18). Here, too, there are maxima for the corresponding ratios, which will be denoted by C_r (0.214), C_θ (0.153), and C_z (0.287). It is possible to make a check on the values of the coefficients C_r, C_θ, and C_z obtained independently. Indeed, as noted in Sec. 17, the solution in the neighbourhood of a smooth edge has the same asymptotic properties as in plane strain. Hence, there must be a relationship among the coefficients C_r, C_θ, C_z (see Sec. 15), namely

$$C_\theta = \nu (C_r + C_z) \tag{37.2}$$

Note that this equality is fulfilled to within 2 per cent for the values of the coefficients obtained. There is one more relationship among the coefficients. According to (17.8), the eigensolutions for a wedge (which form the asymptotic terms) may be expressed in terms of one constant. In the present case (when $\alpha = 0.75\pi$) we obtain the relation

$$C_z = 1.483\, C_r. \tag{37.3}$$

Here the difference between the exact value and the value obtained by calculation is 9 per cent.

Consider the results of the solution of the interior problem. Here the solution of the boundary value problem itself (and not of the integral equation) is trivial, namely all normal stress components are

equal to unity, and the tangential components are zero. Table 19 gives stress values at points of the axis of revolution at a specified distance from the vertex. Table 20 gives stress values in the middle plane at a specified distance from the edge.

Table 19

ρ	0.002	0.003	0.005	0.020
$\sigma_r = \sigma_\theta$	1.0048	1.011	1.005	0.9956
σ_z	1.0041	1.026	1.040	0.9997

Table 20

ρ	0.0004	0.0007	0.0050	0.020
σ_r	0.7277	0.7424	0.9249	0.9603
σ_θ	0.8775	0.9355	0.9857	1.0062
σ_z	0.9744	0.9673	1.0352	1.0154

It follows from the results obtained that the solution differs from the exact one only in a very small neighbourhood of irregular points.

Note that in the cases under consideration the question of asymptotic properties does not arise since the corresponding equations (see Figs. 7 and 8) have no solutions in the range from 0 to 1 for the angles involved.

In conclusion it may be noted that the method of V. G. Maz'ya and B. A. Plamenevskii, which has been mentioned in Sec. 25, for determining the coefficients of eigenfunctions is automatically extended to the case of a conic point. The same authors [3] have considered the case when the surface has an edge.

38. Mixed (Contact) Problems

Let S_1 be a part of the surface S bounding an elastic body D and let S_2 be the remainder. The closed smooth line that is the boundary between S_1 and S_2 is denoted by L.

The solution of the mixed problem of the theory of elasticity consists in determining the displacement field $U(p)$ in the region D subject to the boundary conditions

$$U(q) = f_1(q) \quad (q \in S_1), \quad T_n U(q) = f_2(q) \quad (q \in S_2). \tag{38.1}$$

Problems of this kind are called *mixed* (or *contact*). Though these problems are extremely important in applications, they are not adequately studied in general terms because of their complexity. The proof of the existence of the solution to the physical problem is given by G. Fichera [1],* but the construction and investigation of the integral equations for these problems cannot be considered completed.

The integral equations for mixed problems are not difficult to construct. Indeed, if the displacement is sought in the form of a simple layer potential, then, by applying a limiting process and passing to the points of the surface S_1 (for displacements) and to the points of the surface S_2 (for stresses), we arrive at an integral equation with discontinuous coefficients and kernels. The question of the solvability of such equations is left open.

In the work of A. Ya. Aleksandrov [7] it is suggested that the equation in question should be solved by the mechanical quadrature method using the results of this work [1] to evaluate the singular terms. In the work of T. A. Cruse [1] the mechanical quadrature method has been used to solve integral equations obtained by applying a suitable limiting process in Betti's formula (14.27).

Note that the singular integral equations of the second kind with discontinuous kernels can be obtained by using the representation of displacements in the form of the sum of two potentials, namely a simple layer potential (with density on the surface S_2) and a double layer potential (with density on the surface S_1). The consideration of mixed problems on the basis of Green's matrix may be found in the work of V. D. Kupradze [3].

The approximate method for the solution of the fundamental problems (see V. D. Kupradze and M. A. Aleksidze [1]) is used by M. A. Aleksidze and K. N. Samsoniya [2] for solving mixed problems. The authors construct a set of particular solutions of Lamé's equations, $\Gamma(p, p_k)\,\varphi(p_k)$, using a set of points p_k situated outside the region D. The corresponding coefficients $\varphi(p_k)$ are found by the method of least squares.

It is also possible to implement the following computational scheme. On the surface S_1 or S_2 (depending on which surface is the smaller) the unknown conditions (stresses for the surface S_1) are represented in some constructive manner, e.g. as a series expansion (introducing a certain co-ordinate system). For each harmonic, the solution of the

* For a harmonic problem, the proof has been obtained by S. Zaremba [1].

corresponding boundary value problem must be constructed, after which the unknown coefficients are determined from the condition of minimum mean discrepancy for the known boundary conditions on the surface under consideration.

This method appears to have considerable promise in problems where the tangential stress component and the normal displacement component are known on the surface S_1 since in this case the unknown is a scalar, and not a vector, function. It is precisely this case that is

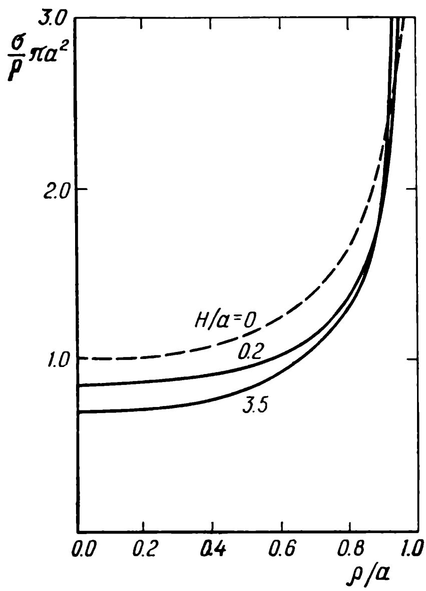

Fig. 28. Contact stresses for different penetrations (H/a)
Dashed line shows contact stresses for $H = 0$ (exact solution)

considered in the work of V. M. Likhovtsev and P. I. Perlin (see V. M. Likhovtsev [1]), with reference to the problem for a half-space with a cylindrical cutout. The force P is transmitted through a smooth flat punch occupying the whole end. The normal stress component is taken in the form of a series, namely

$$\sigma(\rho/a) = [1-(\rho/a)^2]^{-2/3}\left[\sum_{n=0}^{\infty}\alpha_n(\rho/a)^{2n}\right].$$

The structure of the multiplying factor follows from Eq. (17.7) when $\alpha = 1.5\,\pi$. The calculations are carried out retaining one or two coefficients for different values of the ratio of the height H of the cylinder to its radius a. Figure 28 shows contact stress diagrams for $H/a = 0.2$ and 3.5. The exact solution for $H = 0$ is shown by a dashed line.

Table 21 gives values of the coefficients α_n (with a different number of terms retained in the series), the corresponding error in displacement, and the contact stresses $\sigma' = \pi a^2 \sigma (\rho/a)/P$ $(H/a = 0.2)$.

Table 21

Coefficients of the Function $\sigma(\rho/a)$, Error in Displacement, and Contact Stresses with a Different Number of Terms Retained in the Series

α_0	α_1	α_2	Δ (%)	σ' $(\rho/a = 0.0)$	σ' $(\rho/a = 0.5)$	σ' $(\rho/a = 0.9)$
1.000	—	—	4	0.678	0.822	2.038
1.276	−0.357	—	1	0.866	0.972	2.010
1.266	−0.408	0.120	0.6	0.856	0.960	2.005

We now turn to the consideration of problems for some specific regions. Naturally, we shall be concerned with specialized methods taking account of the structure of these regions. Contact problems for a half-space appear to be the most studied.

In cases where all displacement components are given on the surface S_1 (so-called cohesion with the punch) the integral equation can be obtained from Cerruti's solution (see W. Nowacki [1]) for a concentrated force applied to the boundary of a half-space. By summing the displacements due to all three components, we can obtain a representation of displacements on the surface S_1 in integral form.

Let us discuss in greater detail the most studied case when the shearing stresses are absent, using for this purpose Boussinesq's solution. According to this solution, the displacement component normal to the boundary due to the action of a concentrated force P applied at a point q takes an extremely simple form on the surface:

$$U = \frac{P(1-\nu^2)}{\pi E} \frac{1}{r(p, q)} = \alpha \frac{P}{r}.$$

The integral equation following immediately from this formula is

$$\alpha \int\limits_{S_1} \frac{p(q')}{r(q, q')} dS_{q'} = f_1(q) \quad (q \in S_1). \tag{38.2}$$

We shall mention the well-known analogy between the mixed problem for a half-space in the absence of shearing stresses on its surface and the problem for a space with the same plane cut loaded only by normal pressure (equal on both sides). Since there is a plane of sym-

metry, the latter problem may be regarded as a mixed one when pressures are given on the surface of the cut and zero normal displacements outside the cut. By superimposing the solution of the second fundamental problem for a half-space with the same stress values on the surface S_1, we arrive at a mixed problem when the stresses on the surface S_1 are zero and displacements are given outside S_1.

Thus, the solution of the mixed problem of elasticity theory is reduced to the solution of the mixed problem of potential theory, which (by symmetry) is equivalent to the Dirichlet problem for the whole space when a value of the function is given on the surface S_1. According to F. Tricomi [1], the solution of the last problem can be sought in the form of a simple layer potential, which leads to the same equation (38.2).

It follows from the results of S. Zaremba [1] that there exists a harmonic function representable in the form of a simple layer potential (naturally, if the function $f(q)$ is sufficiently smooth). Equation (38.2) is therefore solvable, and moreover in a unique manner. S. G. Mikhlin [4] indicates the possibility of applying to the integral equation (38.2) (as to an equation with a symmetrical kernel) of the Hilbert-Schmidt theorem concerning the representability of the solution as a series in eigenfunctions. True, we can speak only of the convergence in the mean. The implementation of this approach has been illustrated by V. I. Dovnorovich (see S. G. Mikhlin [4]) by considering an axially symmetric problem.

A large number of works are devoted to the derivation of some particular results for the solution of Eq. (38.2), and an account of many of these works dealing with the cases of a circular and an elliptical punch may be found in the monographs of I. Ya Shtaerman [1], L. A. Galin [1] and A. I. Lur'e [1].

Let us now consider approximate methods of solving Eq. (38.2) when the surface S_1 is close (in a sense) to a circle. In the work of A. B. Efimov and V. N. Vorob'ev [3] it is suggested that the region S_1 should be mapped into a circle by introducing a certain parameter, which is considered to be sufficiently small, into the mapping function. The integral equation is accordingly transformed (the kernel is represented as a series in terms of the parameter). Let us write this equation symbolically in the form $\boldsymbol{K}p = f$ and represent it as

$$\boldsymbol{K}_0 p = f - (\boldsymbol{K} - \boldsymbol{K}_0)\, p, \tag{38.3}$$

where the operator $\boldsymbol{K}_0$ corresponds to a zero value of the parameter (i.e., when the surface S_1 is a circle, and therefore the inverse operator $\boldsymbol{K}_0^{-1}$ is known). Equation (38.3) is solved by the method of successive approximations, which leads to recurrence relations:

$$p_n = p_0 - \boldsymbol{K}_0^{-1} (\boldsymbol{K} - \boldsymbol{K}_0)\, p_{n-1} \quad (n = 1, 2, \ldots)$$

It is obvious that if the value of the parameter is sufficiently small, the process converges. Note that in the case of a flat elliptical punch the process converges for all values of the eccentricity.

We shall also mention a procedure suggested by M. Ya. Leonov and K. T. Chumak [1]. The authors divide the region S_1 into an inscribed circle and the remaining annular region. The stresses in the annular region are represented in analytic form (by using local co-ordinates in the direction of the normal (ρ) and the tangent (s) to the contour L): $p(\rho,s) = \beta(s)\rho^{-1/2}$. The factor $\beta(s)$ is determined as follows. The contact problem is solved for a circular punch with the load p (ρ, s), and then an equation for the function β (s) is obtained from the conditi n that the stresses on the contour of the circle should be finite. This scheme can be improved if it is assumed that the coefficient β (s) depends linearly (or in a more complex manner) on the co-ordinate ρ.

Let us consider the solution of the contact problem for a sphere with axial symmetry (see V. F. Bondareva [2]). In this work the author uses the previously obtained expression (see V. F. Bondareva [1]) for the radial component of displacement under axially symmetric normal pressure. As a result, the author arrives at an integral equation of the first kind on a spherical segment S_1. It is possible to separate terms in the kernel of the integral (by a suitable change of the variables) that are identical with the terms in the equation for the axially symmetric contact problem for a half-space (see M. Ya. Leonov [2]) (in the same work the solution of this equation is obtained in closed form). The separated terms are transposed and all the other terms are considered as the right-hand side of an auxiliary equation whose solution leads to an integral equation of the second kind with a regular kernel. It is shown that the resulting equation can be solved by the method of successive approximations.

Conclusion

The material presented in the book demonstrates convincingly the possibilities inherent in the solution of elasticity problems using the integral equation techniques. Because of lack of space and limited scope of the book, the authors, naturally, have left out of consideration some directions where these techniques are also used and which are an obvious generalization of the approaches considered above. Among these are the problems of the theory of periodic vibrations (see, for example, V. D. Kupradze, T. G. Gegelia *et al.* [1]), of the theory of the deformation of anisotropic media (see, for example, S. M. Vogel and F. J. Rizzo [1], M. D. Snyder and T. A. Cruse [1]).

Some methods involving integral equations of the second kind as an intermediate stage also fall out of the authors' view.

The integral transformation methods, which have received much recognition (see, for example, Ya. S. Uflyand [1]), provide a means for finding efficient solutions of problems for regions of a special kind. Mixed problems and problems for regions with cuts can be reduced to integral equations of the first kind with a discontinuous kernel (so-called dual or paired equations), which are reduced to integral equations of the second kind by a special representation of the unknown function. Note that when using paired series equations we arrive at the same equations (see, for example, V. Z. Parton and E. M. Morozov [1]).

A large number of problems are solved by so-called approximate factorization of the functional relation resulting from the application of the integral transformation to a class of integral equations of the first kind (see, for example, B. Noble [1]).

The book does not cover the methods of solving integral equations of the first kind to which some elasticity problems can be reduced without difficulty. As is known (see, for example, A. N. Tikhonov and V. Ya. Arsenin [1]), the solution of these equations by conventional approximate methods leads to unstable algorithms, and therefore recourse is made to special procedures (so-called regularizing algorithms).

The investigators' attention is also attracted by problems for stiffened bodies, i.e. piecewise homogeneous bodies in which the extent of the region with a large elastic modulus is much less than that of the matrix. Hence, it is not advisable to determine the state of stress in these regions using general methods (see Secs. 22, 36). There are approximate procedures available for the solution of problems of this kind when the state of stress in stiffeners is found by the methods of structural mechanics, which enables one to pass to a special auxiliary boundary value problem for the matrix. At this stage the possibility exists of making efficient use of integral equations (see, for example, G. N. Savin and V. I. Tul'chii [1], G. N. Savin and N. P. Fleishman [1]).

It is known that some methods of solving problems of continuum mechanics lead at intermediate stages to problems of linear elasticity. We shall mention the elastic solution method employed in plasticity theory (deformation theory) (see A. A. Il'yushin [1]). In the work of J. L. Swedlow and T. A. Cruse [1] a similar approach is extended to the flow theory. Of course, the solution of these problems can be obtained (under certain restrictions) by the integral equation method, and this has been proposed in the cited work.

It is obvious that the elastic solution method can also be used (in a certain form) for media with more complex rheology.

Of great practical importance are elasticity problems for media

with continuously varying Lamé's coefficients. Note that with some restrictions on the behaviour of these coefficients the solution can be reduced to a consideration of successive auxiliary problems for the homogeneous medium.

Attempts are made to develop certain general approaches to a direct solution of problems for non-homogeneous media on the basis of integral representations (see Furuhashi Rozo and Kataoka Masoharu [1]).

It is hoped that the numerical methods of solving integral equations opening up great possibilities for an efficient study of many problems of continuum mechanics will receive ever increasing attention of investigators.

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